

Exam 19.12.2006

No calculators or texts allowed. Time for the exam is three hours.

1. Let K be a closed convex set in a Hilbert space V . Show that for given $v \in V$ there exists a unique u^* in K such that

$$\|v - u^*\| = \inf_{u \in K} \|v - u\|.$$

How can u^* be characterized in terms of the inner product?

2. Consider the initial-boundary value problem

$$\begin{cases} u_t = u_{xx}, & x \in (0, 1), t > 0 \\ u(0, t) = u(1, t) = 0, & t > 0 \\ u(\cdot, 0) = v. \end{cases}$$

The numerical scheme uses initial values $U_j^0 = v(x_j)$ on a uniform grid $0 = x_0 < x_1 < \dots < x_M = 1$ and $U_0^n = U_M^n$, $n > 0$. We will use a combination of the backward Euler scheme

$$\bar{\partial}_t U_j^{n+1} = \partial_x \bar{\partial}_x U_j^{n+1}$$

and the Crank-Nicolson (implicit midpoint) scheme

$$\bar{\partial}_t U_j^{n+1} = \frac{1}{2} \partial_x \bar{\partial}_x (U_j^n + U_j^{n+1}).$$

What will be gained (in comparison to the pure Crank-Nicolson scheme) by using mainly the latter and every now and then a backward Euler step? Hint: consider the rapidly decreasing modes of the space discretized equation.

Assume the total number of backward Euler steps will be bounded as the time and space discretization parameters tend to zero. Show that the overall method is of second order. You can assume that the stability of both of the methods are already shown.

3. Consider numerically solving the problem

$$\begin{cases} u_t = au_x, & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u(\cdot, 0) = v, & \text{in } \mathbb{R}, \end{cases}$$

where a is a constant. Under what condition the Friedrichs scheme

$$U_j^{n+1} = \frac{1}{2}((1 + a\lambda)U_{j+1}^n + (1 - a\lambda)U_{j-1}^n),$$

$$U_j^0 = v(x_j),$$

is stable in the 2-norm?

4. State the CFL (Courant-Friedrichs-Levy) necessary stability condition. No proof is needed. Define also the main concepts you use.