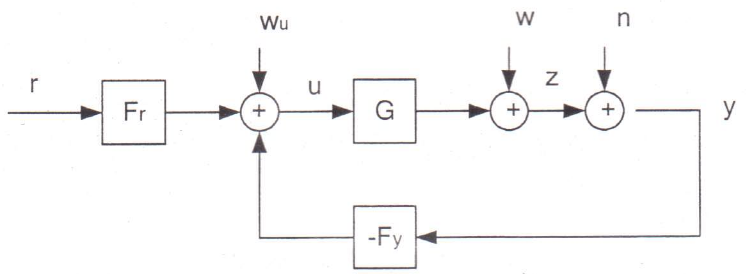
AS-74. 3123 Model-Based Control Systems
Exam 17. 12. 2007

The questions are available only in English. You can answer in Finnish, Swedish or English. The final grade is given when both the examination and the homework problem have been evaluated and accepted.

5 problems.

1. Consider the control configuration shown in the figure.



Let $G(s) = \dfrac{s-1}{s+1}$ and $F_y = F_r = \dfrac{1}{s-1}$.

a. Determine the *loop transfer function*, *sensitivity function* and *complementary sensitivity function* to the system. Draw schematically the gains of the frequency responses (magnitude plots as Bode diagrams).

b. Show analytically that the system is not internally stable.

2. Consider the scalar system

$$\dot{x}(t) = 2x(t) + 2u(t) + w(t)$$
$$y(t) = 3x(t) + v(t)$$

in which the noise terms w and v are white noise with intensity 1 and 5, respectively.

a. Form the (stationary) Riccati equation and solve it.

b. Calculate the Kalman-gain and form the (stationary) Kalman-filter. Calculate the poles of the estimator.

c. How do the poles move, if the intensity of the measurement noise grows tenfold? Explain the result.

3.a. Let $G(s)$ be a transfer function of a SISO-system, $L(s)$ the loop transfer function and $S(s)$, $T(s)$ correspondingly the sensitivity and complementary sensitivity functions. Prove the following so-called *interpolation constraints*

$$G(z) = 0 \Rightarrow L(z) = 0 \Leftrightarrow T(z) = 0, S(z) = 1$$
$$G^{-1}(p) = 0 \Rightarrow L(p) = \infty \Leftrightarrow T(p) = 1, S(p) = 0$$

where z and p are the RHP-zero and RHP-pole of the transfer function $G(s)$.

3. b. Consider a SISO system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with the state feedback control law $u(t) = -Lx(t)$ corresponding to a well-defined LQ problem. Show that the *loop gain function*, when calculated from the **input side** of the process is

$$H(s) = L(sI - A)^{-1} B$$

3. c. Consider the loop gain function $H(s)$ shown in problem 3b. According to a known lemma it holds that

$$|1 + H(-j\omega)||1 + H(j\omega)| \geq 1$$

Prove that for any LQ controlled system the phase margin is at least 60 degrees and the gain margin is infinite.

4. Consider the multivariable system

$$G(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{s+1} & 0 \end{bmatrix}$$

a. Calculate the Relative Gain Array (RGA) at zero frequency. Based on the result, what can be deduced concerning the controllability of the system?
b. Calculate the singular values of the system at zero frequency. Explain the meaning of the singular values in this context.
c. Design a diagonalizing compensator for $G(s)$.

5. Explain briefly the following concepts:
- Small-Gain Theorem,
- Fundamental limitations in control,
- Loop-shaping control

Some formulas, which might be useful:

$$\dot{x} = Ax + Bu + Nv_1$$
$$y = Cx + Du + v_2$$

$$\dot{x} = Ax + Bu, \quad t \geq t_0$$

$$J(t_0) = \frac{1}{2} x^T(T)S(T)x(T) + \frac{1}{2}\int_{t_0}^{T} \left(x^T Qx + u^T Ru\right) \; dt$$

$$S(T) \geq 0, \quad Q \geq 0, \quad R > 0$$

$$-\dot{S}(t) = A^T S + SA - SBR^{-1}B^T S + Q, \; t \leq T, \text{ boundary condition } S(T)$$

$$K = R^{-1}B^T S$$

$$u = -Kx$$

$$J^*(t_0) = \frac{1}{2} x^T(t_0)S(t_0)x(t_0)$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K\left(y(t) - C\hat{x}(t) - Du(t)\right)$$
$$K = (PC^T + NR_{12})R_2^{-1}$$
$$AP + PA^T - \left(PC^T + NR_{12}\right)R_2^{-1}\left(PC^T + NR_{12}\right)^T + NR_1N^T = 0$$

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t) + Nv_1(t) - Kv_2(t)$$