AS-74. 3123 Model-Based Control Systems
Exam 21. 12. 2006

The questions are available only in English. You can answer in Finnish, Swedish or English. The final grade is given when both the examination and the homework problem have been accepted.

5 problems.

1. Explain briefly the following concepts

- Principle of Optimality
- Dynamic programming
- Waterbed effect
- Robust stability
- Small gain theorem
- Internal model control

2.a. Draw a schema of the "one-degree-of-freedom" control configuration. Define the concepts *sensitivity function* and *complementary sensitivity function* for it.

2.b. Derive the following relationships (in the formulas *S* and *T* are the sensitivity and complementary sensitivity functions, and *L* is the open loop transfer function matrix; the dimensions are assumed to be appropriate)

$$S(i\omega) + T(i\omega) = I$$
$$|S(i\omega)| + |T(i\omega)| \geq 1$$
$$T(i\omega) = (I + L(i\omega)^{-1})^{-1}$$

2.c. Consider a SISO-case. Determine the region in the complex plane where $|S| > 1$. How can the result be explained in view of control performance?

3. Consider a SISO-process with the transfer function

$$G(s) = \frac{s + \dfrac{1}{T_1}}{\left(s + \dfrac{1}{T_2}\right)\left(s + \dfrac{1}{T_3}\right)}, \quad T_1 < 0, \quad T_2 < 0, \quad T_3 > 0$$

Explain, what kind of fundamental limitations on control performance can be stated for this system? (Present also some calculations; the formulas in the end of the problems can be of help.)

4. a. Given the system

$$\dot{x}(t) = -2x(t) + 3v(t)$$
$$y(t) = 4x(t)$$

where v is white noise with a spectral density 16. Calculate the steady-state covariance of x and y.

4.b. Let A be a non-singular mxm-dimensional matrix. Prove that for the singular values it holds

$$\bar{\sigma}(A^{-1}) = 1/\underline{\sigma}(A) .$$

Hint: Use the singular value decomposition (SVD).

5. Consider the system

$$\dot{x}_1 = -x_1 + u$$
$$\dot{x}_2 = x_1$$

Determine a control law that minimizes the criterion

$$J = \int_0^\infty (x_2^2 + 0.1u^2)dt$$

(In the above equations the time symbol t has been dropped for brevity.)

Some formulas, which might be useful:

$$\int_0^\infty \log |S(i\omega)| d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}\ (\mathrm{p_i})$$

$$|W_T(p_1)| \le 1 \quad \Rightarrow \quad \omega_0 \ge \frac{p_1}{1 - 1/T_0}$$

$$|W_S(z)| \le 1 \quad \Rightarrow \quad \omega_0 \le (1 - 1/S_0)\, z$$

$$A\Xi_x + \Xi_x A^T + NRN^T = 0$$

$$\dot{x} = Ax + Bu, \quad t \geq t_0$$

$$J(t_0) = \frac{1}{2} x^T(T) S(T) x(T) + \frac{1}{2} \int_{t_0}^{T} \left(x^T Q x + u^T R u \right) \; dt$$

$$S(T) \geq 0, \quad Q \geq 0, \quad R > 0$$

$-\dot{S}(t) = A^T S + SA - SBR^{-1} B^T S + Q, \; t \leq T,$ boundary condition $S(T)$

$$K = R^{-1} B^T S$$

$$u = -Kx$$

$$J^*(t_0) = \frac{1}{2} x^T(t_0) S(t_0) x(t_0)$$