Helsinki University of Technology
Department of Mathematics

Sofieva

Mat-1.1632 Mathematics 3-II

Please fill in the required information into each paper sheet.
Only ordinary calculators are allowed.

1. Solve the system $\begin{cases} \dot{y}_1 = 4y_2 \\ \dot{y}_2 = 4y_1 - 4 \end{cases}$ (6p)

2. Find critical points of the system and determine their type and stability

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^2 \end{cases}$$ (6p)

3. (a) Does every continuous function have a Laplace transform? Give a reason or a counterexample. (1p)
(b) Solve the initial value problem by the Laplace transform: $y' - y = 1,\ y(0) = -1$ (5p)

4.(a) Does the Fourier sine series of the function $f(x) = \cos^2 x$ exist? (2p)
(b) Show that if $f(x)$ has a Fourier transform so does $f(x-a)$, a =const. (2p)
(c) Give reasons why $f(x)=1$ has neither a Fourier cosine transform nor a Fourier sine transform. (2p)

5. Consider the mixed problem for heat equation

(1) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$
(2) $u(x,0)=f(x)$
(3) $u(0,t)=0,\ u(1,t)=0$

(a) Substituting $u(x,t)=X(x)T(t)$, derive the ordinary differential equations $X'' + \lambda^2 X = 0$ and $T' + \lambda^2 T = 0$, where $\lambda \in \mathbf{R}$ is a constant (1p)
(b) Find the function $X(x)$ satisfying the boundary conditions derived from Eq.(3) and the constant λ (1p)
(c) Show that the solution to the problem (1), (3) can be presented in the form

(4) $$u(x,t) = \sum_{k=1}^{\infty} B_k \exp(-\pi^2 k^2 t)\sin(\pi k x)$$ (1p)

(d) Then find the solution that satisfies (2) (Find coefficients B_k). (1p)
(e) Find the solution to the problem (1), (2), (3), if $f(x)=2\sin \pi x + 3 \sin 2\pi x$ (2p)

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Appendix A. Table of Laplace transforms. $L(f(t)) = F(s) = \int_0^{\infty} f(t)e^{-st}dt$

№	f(t)	F(s)	№	f(t)	F(s)
1)	1	$\frac{1}{s}$	6)	$e^{\alpha t}\cos \beta t$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$
2)	$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$	7)	$e^{\alpha t}\sin \beta t$	$\frac{\beta}{(s-\alpha)^2+\beta^2}$
3)	$e^{\alpha t}$	$\frac{1}{s-\alpha}$	8)	$\frac{t^n}{n!}e^{\alpha t}$	$\frac{1}{(s-\alpha)^{n+1}}$
4)	cos βt	$\frac{s}{s^2+\beta^2}$	9)	cosh βt	$\frac{s}{s^2-\beta^2}$
5)	sin βt	$\frac{\beta}{s^2+\beta^2}$	10)	sinh βt	$\frac{\beta}{s^2-\beta^2}$

Appendix B. Fourier series
Any periodic (with period 2L) piecewise continuous in the interval $-L \le x \le L$ function $f(x)$ can be represented by the Fourier series

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{\pi n x}{L} + B_n \sin\frac{\pi n x}{L}\right),$$

where $$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{\pi n x}{L}dx \quad , n=0,1,2\ldots$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{\pi n x}{L}dx \quad , n=1,2\ldots$$

Appendix C. Fourier transform

The Fourier transform $F(w)$ of a function $f(x)$ is $$F(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-iwx}dx$$