Institute of mathematics
Helsinki University of Technology

Arponen/Baarman

Mat-1.3651 Matrix Computations (Numeerinen matriisilaskenta)

Final examination 09.05.2009

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination. Study programmes are ARK, AUT, BIO, EST, ENE, GMA, INF, KEM, KJO, KTA, KON, MAK, MAR, PUU, RAK, TFY, TIK, TLT, TUO, YHD.

Calculators are not allowed nor needed. Time for the exam is 4 hours. You can answer in Finnish if you wish.

1. Let
$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 8 & 4 & 8 \\ 6 & 7 & 9 \end{pmatrix}.$$
Use the Gaussian elimination with partial pivoting to compute $\mathbf{\Pi}, \mathbf{L}$, and $\mathbf{U}$ s.t. $\mathbf{\Pi A} = \mathbf{LU}$ where $\mathbf{L}$ is unit lower triangular, $\mathbf{U}$ upper triangular and $\mathbf{\Pi}$ a permutation. (Hint: $\mathbf{U}$ should be integers now!)

2. Let $\mathbf{A} \in \mathbb{C}^{m\times m}$. Show that

   (a) there exist (column) vectors $\mathbf{u}_j, \mathbf{v}_j \in \mathbb{C}^m$, $j = 1, \dots, m$ such that
$$\mathbf{I} - z\mathbf{A} = (\mathbf{I} - z\mathbf{u}_m\mathbf{v}_m^*)\cdots(\mathbf{I} - z\mathbf{u}_2\mathbf{v}_2^*)(\mathbf{I} - z\mathbf{u}_1\mathbf{v}_1^*) \quad \forall z \in \mathbb{C}.$$

   (b) The inner products $\mathbf{u}_j^*\mathbf{v}_j$ are the eigenvalues of $\mathbf{A}$.

   Hint: Schur.

3. Show that for any matrix norm, $\|\cdot\|_*$, holds:
$$\|\mathbf{A}\|_* \geq \rho(\mathbf{A}) = \max_{\lambda\in\Lambda(\mathbf{A})} |\lambda| = \lim_{k\to\infty} \|\mathbf{A}^k\|_*^{1/k}.$$

4. Show that for every $\mathbf{A} \in \mathbb{C}^{m\times n}$ there exists a Hermitian positive semidefinite $\mathbf{P}$ and $\mathbf{U} \in \mathbb{C}^{m\times n}$, with $\mathbf{U}^*\mathbf{U} = \mathbf{I}$ such that $\mathbf{A} = \mathbf{PU}$ and $\mathbf{P}^2 = \mathbf{AA}^*$.

5. Describe

   (a) what is the Arnoldi iteration, and

   (b) how is it used in the GMRES iteration.