Tfy-0.3223 Statistical Physics Intermediate exam / välikoe II, Wed May 13, 2009

Pick four (4) problems.

1. *Ising model in one dimension*: consider a ring of Ising spins with N spins of value ± 1. The Ising Hamiltonian is $H = -\sum_{i,i+1} J\sigma_i\sigma_{i+1} - h\sum_i \sigma_i$, where h is the external field. Spin N is coupled to spin 1.

(a) Write down the transfer matrix T for $h = 0$.

(b) Compute the Hemholtz free energy in the thermodynamic limit $N \to \infty$.

(c) As in (b), but now compute the correlation length.

2. *Bond percolation on Cayley trees*: Consider bond percolation on a hierarchical tree-like lattice in which each site at one level is connected to z sites at the level below. Thus, the n-th level of the tree has z^n sites.

(a) For $z = 2$, find a recursion relation for the probability $P_n(p)$ that the top site of the tree is connected to some site at the bottom level.

(b) Find the limiting behavior of $P(p)$ for infinitely many levels. The critical point p_c is the point for which $P > 0$ for $p > p_c$. Find p_c.

(c) The (order parameter) critical exponent β is defined by $P \propto (p - p_c)^\beta$ near the critical point. Find β for the case above.

(d) Obtain an implicit equation for a tree with branching number z as you did in part (b). Solve it for small P, i.e. in the vicinity of p_c. Find p_c and β.

(e) Find the probability $P_n(p)$ in the case $z = 1$, and deduce the correlation length ξ as a function of p.

3. *Langevin equation:* Start from the Langevin equation

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\frac{v(t)}{\tau} + A(t),$$

for a particle with mass m in one dimension. Above, $A(t)$ is the instantaneous acceleration caused by a random force, and it satisfies $\langle A(t)\rangle = 0$ and $\langle A(t)r(t)\rangle = 0$, where $r(t)$ is the position of the particle, since the random force is expected to be independent of position. Derive a second-order differential equation with respect to time for $\langle r^2\rangle$ in the thermodynamical equilibrium, and solve it using the initial conditions $\langle r^2(t = 0)\rangle = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\langle r^2(t)\rangle|_{t=0} = 0$. (Hint: multiply both sides by r, and use the equipartition theorem where applicable.) How does the solution behave when $t \ll \tau$? How does it behave when $t \gg \tau$?

TURN

4. *Fluctuations and dissipation:*

(a) What does the fluctuation-dissipation (FD) theorem state as derived for Brownian motion?

(b) What are the main steps for the derivation?

(c) Pick either the Crooks or Jarzynski equality. Explain the relation of this to the FD theorem, concisely.

5. *Critical phenomena*

(a) The (saddle-point) free energy of the Ginzburg-Landau Hamiltonian is given by the relation $\beta F_{sp} \simeq \beta F_0 + V \min_{\vec{m}} \psi(\vec{m})$ where $\psi(\vec{m}) = (t/2)m^2 + um^4 + \cdots - \vec{h} \cdot \vec{m}$, $u = u_0$, $t = a_0 + a_1(T - T_c)$. Derive the magnetization and specific heat as a function of temperature ($h = 0$). What kind of critical exponents do you get?

(b) The theory of homogeneous functions says that $f_{sing}(t, h) = |t|^{2-\alpha} g_f(h/|t|^{\Delta})$. The thing on the left-hand side is the singular part (close to the phase transition) of the free energy, and t is reduced temperature (and h the magnetic field). Explain the right-hand side. What are the exponents, what does g_f do as a function of its argument?.

(c) Renormalization: let us consider the behavior of free energy. In rescaling, $V' f(t', h')) = VF(t, h)$ where V is the original volume, and if we rescale $x' = x/b$ then $f(t, h) = b^{-d} f(b^{y_t} t, b^{y_h} h)$. Here y_x is the scaling dimension of quantity x. Derive the value of the gap exponent and the specific heat exponent from this as a function of the y_x.