Helsinki University of Technology
Department of Mathematics and Systems Analysis
Mat-1.3610 Extremes of Random Phenomena

Examination
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Examination

1. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with cdf F such that $F(x) = 1 - e^{-x}$ for $x > 0$. Find out the limiting distribution of $(M_n - d_n)/c_n$, where $M_n = \max(X_1, \ldots, X_n)$, $c_n = 1$, and $d_n = \ln n$. (Hint: Recall that $\lim_{n\to\infty}(1 + s/n)^n = e^s$.)

2. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with cumulative distribution function F, and denote the running maximum by $M_n = \max(X_1, \ldots, X_n)$. What can you say about $\lim_{n\to\infty} M_n$? Does the limit exist?

3. Let $X_1, \ldots, X_n$ be independent random variables distributed according to the standard generalized extreme value distribution H_ξ with $\xi > 0$.

(a) Show that $M_n = \max(X_1, \ldots, X_n)$ follows a generalized extreme value distribution.

(b) Determine constants μ, σ, and α so that M_n has the same distribution as $\mu + \sigma Z_\alpha$, where Z_α has the standard Fréchet distribution with shape α.

4. Table 1 represents $n = 3$ observations of annual maximum temperatures measured in an imaginary land called Extremistan.

(a) Compute the empirical cumulative distribution function F_n of the observed temperatures.

(b) Assume that the future annual maximum temperatures $X_1, X_2, \ldots$ are independent and identically distributed random variables with cdf F_n, and denote $L(u) = \min\{i \geq 1 : X_i > u\}$. Compute the probability $\mathrm{P}(L(u) = 3)$ for $u = 50$.

(c) Compute the return period $\mathrm{E}\, L(u)$ for $u = 99$.

year	°C
2006	1
2007	1
2008	100

Table 1: Annual maximum temperatures in Extremistan during 2006–2008.

5. (a) Explain why the standard Gumbel distribution is relevant in extreme values statistics.

(b) Give an example of a probability distribution that belongs to the maximal domain of attraction of a Gumbel distribution.

(c) Give an example of a probability distribution that does *not* belong to the maximal domain of attraction of a Gumbel distribution.

Formulas

Standard extreme value distribution functions:

- $\Phi_\alpha(x) = \exp(-x^{-\alpha})$, $x > 0$,
- $\Lambda(x) = \exp(-e^{-x})$,
- $\Psi_\alpha(x) = \exp(-(-x)^\alpha)$, $x \leq 0$.

Standard generalized extreme value distribution function:

$$H_\xi(x) = \begin{cases} \exp\left\{-(1+\xi x)^{-1/\xi}\right\}, & 1 + \xi x > 0, \quad \xi \neq 0 \\ \exp\left\{-e^{-x}\right\}, & \xi = 0. \end{cases}$$

Standard generalized Pareto distribution function:

$$G_\xi(x) = \begin{cases} 1 - (1+\xi x)^{-1/\xi}, & x \geq 0, \quad \xi > 0, \\ 1 - e^{-x}, & x \geq 0, \quad \xi = 0, \\ 1 - (1+\xi x)^{-1/\xi}, & 0 \leq x \leq -1/\xi, \quad \xi < 0. \end{cases}$$