TKK, Department of Mathematics and Systems Analysis Gripenberg
Mat-1.3601 Johdatus stokastiikkaan
Second midterm and final exam 11.5.2009

Write your name, student number, and other information on every paper!
A calculator is allowed in this exam!

Final exams: Problems 1-5
2^{nd} midtrem exam: Problems 3-6
You may do both exams in which case the better result counts!

1. The random variable X is known to be such that $\mathrm{E}(X^2) = 2$. Show that $\mathbb{P}\{X \in [2,3]\} \leq \frac{1}{2}$. Determine also the largest possible lower bound for the probability $\mathbb{P}\{X \in [2,3]\}$ under these assumptions.

2. Give an example of an $\mathbb{R}^2$-values random variable (X, Y) such that X and Y are normally distributed but (X, Y) is not normally distributed, i.e., Gaussian.

3. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables such that $\mathrm{E}(X_j) = 1, \quad j \geq 1$. What can be said about the limit $\lim_{n\to\infty} \frac{\sum_{j=1}^{n} X_j}{\sum_{j=1}^{2n} X_j}$? Give reasons for your answer!

4. Let $X_1, X_2, \ldots$ be independent random variables such that $\mathbb{P}\{X_j = 0\} = 1 - \frac{1}{j^2}$ and $\mathbb{P}\{X_j = j\} = \mathbb{P}\{X_j = -j\} = \frac{1}{2j^2}$, $j \geq 1$. Show that $\mathrm{E}(X_j) = 0$ and $\mathrm{E}(X_j^2) = 1$, $j \geq 1$ but that $\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j$ does not converge in the distribution sense to an $N(0,1)$-distributed random variable when $n \to \infty$.
Hint: What can one say about the series $\sum_{j=1}^{n} \mathbb{P}\{X_j \neq 0\}$ and what does it imply?

5. Let $X_1, X_2, \ldots$ be independent identically distributed random variables so that $\mathrm{E}(X_j) = 1$ for all $j \geq 1$ and let $Y_0 = 1$, $Y_n = \Pi_{j=1}^{n} X_j$ when $n \geq 1$. Is the sequence $(Y_n)_{n=0}^{\infty}$ a martingale with respect to the sequence $(\sigma(X_1, \ldots, X_n))_{n=0}^{\infty}$ (where $\sigma(X_1, \ldots, X_0) = \{\emptyset, \Omega\}$). Give reasons for your answer!

6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $A \in \mathcal{F}$ and let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra in Ω as well. Show that if $\mathbb{P}(A|\mathcal{G}) = \mathrm{E}(1_A|\mathcal{G}) =_{mv} \mathbb{P}(A)$ then $\{A\}$ and $\mathcal{G}$ are independent (from which it then follows that $\sigma(A)$ and $\mathcal{G}$ are independent).

KÄÄNNÄ!