S-88.4200 Statistical Signal Processing.
Final Exam May 21, 2010

1. Define or explain briefly the following concepts:
   (a) Central Limit Theorem
   (b) Cramér Rao Lower Bound
   (c) Consistency
   (d) Influence Function
   (e) Prior distribution
   (f) Sufficient Statistic

2. Kalman Filter
   (a) What problem does the Kalman Filter solve?
   (b) Describe the main steps in the Kalman Filter.
   (c) What is the basic difference between the Kalman Filter and the Extended Kalman Filter?

3. Experiment: Test the "fairness" of a coin.
   Suppose we want to check how biased a coin is. For this purpose we flip the coin N times. We assume that the coin tosses yield N independent identical distributed samples of a Bernoulli distributed value $x_1, \cdots, x_N \in \{\mathrm{F}, \mathrm{T}\}$, where F denotes face and T tail of the coin. Denote the probability that a coin toss yields F with $\Pr(x_n = \mathrm{F}) = p$ and find the maximum likelihood estimate of p!

4. MAP - Estimate
   Suppose N i.i.d. samples x_n having the conditional PDF
   $$f(x_n|a) = \frac{1}{2\sqrt{a}} \mathrm{e}^{-\frac{|x_n|}{\sqrt{a}}}$$
   are observed in an experiment. Furthermore, the prior PDF of a is known to be
   $$f(a) = \begin{cases} a & 1 \leq a \leq \sqrt{3} \\ 0 & \text{otherwise} \end{cases}.$$
   Determine the MAP-estimate for a using all N samples!

5. Cramér-Rao Lower Bound

(a) Suppose *one* realization x of a Rayleigh distributed random variable

$$\mathcal{X} \sim p(x) = \frac{x}{\sigma^2} \mathrm{e}^{-\frac{x^2}{2\sigma^2}},$$

with $x > 0$ is observed.
Derive the Cramér-Rao Bound for the parameter σ!
Hints:

- The Fisher Information matrix is defined as:

$$\mathcal{I}(\theta) = -\mathrm{E}_x \left\{ \frac{\partial^2}{\partial \theta^2} L(x|\theta) \right\} = \mathrm{E}_x \left\{ \left[\frac{\partial}{\partial \theta} L(x|\theta) \right]^2 \right\},$$

with $L(x|\theta) = \log(p(x|\theta))$.

- A usefull integral

$$\int_0^\infty x^n \mathrm{e}^{-ax^2} = \frac{k!}{2a^{k+1}},$$

with $n = 2k + 1$.

(b) Now suppose N independent indentical distributed samples x_n having the same PDF as in (a)

$$p(x_n) = \frac{x_n}{\sigma^2} \mathrm{e}^{-\frac{x_n^2}{2\sigma^2}}$$

are observed.
Derive the Cramér-Rao Bound for the parameter σ given those N i.i.d. samples!