Helsinki University of Technology
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EXAM
7.5.2008

S-72.2410 Information Theory

1. (6p.) Entropy. Three coins are at the bottom of a very murky fountain. Coin A has two heads, coin B has two tails, and coin F is a fair coin with one head and one tail. One of the three coins, selected randomly with equal probability, is taken from the fountain and tossed twice. Let X be the random variable that specifies which coin (A, B, or F) is selected. Let Y_1 and Y_2 be the outcomes (H or T) of the two coin tosses, and let Z be the number of heads (0, 1, or 2) in the two tosses.

   (a) Find $H(X, Y_2)$.

   (b) Find $H(X|Z)$.

   (c) Find $I(Y_1; Y_2)$.

2. (6p.) Source coding. A source has an alphabet of 4 letters. The probabilities of the letters and two possible sets of binary codewords for the source are given below:

Letter	Probability	Code I	Code II
a_1	0.4	1	1
a_2	0.3	01	10
a_3	0.2	001	100
a_4	0.1	000	1000

   For *each* code, answer the following questions (with motivations).

   (a) Does the code satisfy the prefix condition? Is the code uniquely decodable?

   (b) What is the mutual information of the source letter and the first bit of the codeword?

3. (6p.) Channel capacity. What are the capacities of the channels in Figure 1? No motivations are needed. You might want to use the fact that the Z-channel with crossover probability p is known to have a capacity of $\log_2\left(1 + (1-p)p^{p/(1-p)}\right)$ bits per transmission (you need not prove this). The Z-channel is a channel with transition probabilities given by

$$Q = \begin{bmatrix} 1 & 0 \\ p & 1-p \end{bmatrix}.$$

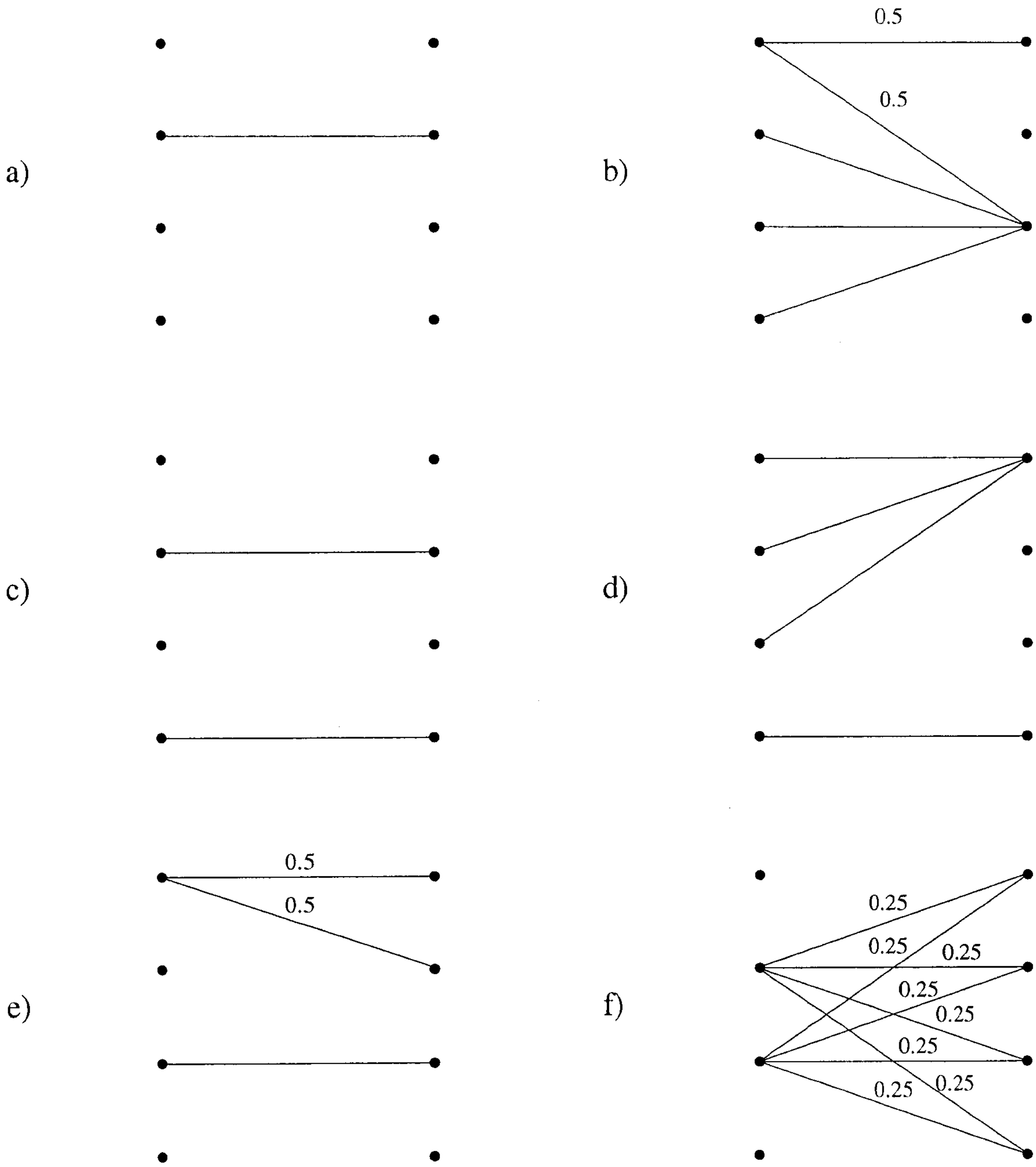

Figure 1: Channels

4. (6p.) Universal and Huffman codes.

   (a) *Briefly* explain the concept of universal source coding, and compare the main characteristics of universal codes and Huffman codes.

   (b) (3p.) Consider a source with an alphabet of size $m \geq 3$ and a probability distribution $p_1, p_2, \ldots, p_m$ satisfying $p_1 \geq p_2 \geq \cdots \geq p_m$. Prove that for any binary Huffman code, if the smallest probability is $p_m < \frac{1}{3}$, then the codeword assigned to this symbol is at least of length 2. Show that the bound $p_m < \frac{1}{3}$ cannot be improved, that is, find a probability distribution with $p_m = \frac{1}{3}$ that has a Huffman code with a codeword of length 1 assigned to this symbol.