T-61.3040 Statistical Modeling of Signals

Final Exam 6.3.2010

In the exam you are allowed to have a calculator (non-programmable or memory emptied) and basic mathematical tables (no tables containing material directly associated with the course). For example, the book "BETA Mathematics Handbook for Science and Engineering" by Råde & Westergren contains material that is too directly associated with the course; therefore you are NOT allowed to have that book in the exam. The results of the exam will be announced eventually through the Noppa system.

1. (max 6p)

Explain the following topics *briefly but covering the most important properties*:

i) LMS algorithm (2p)

ii) Wold decomposition (2p)

iii) The maximum entropy method (2p)

2. (max 6p)

Consider the complex-valued WSS process $y(n)$. You know the following autocorrelations of the process: $r_y(0) = 3$, $r_y(1) = -i$, $r_y(2) = 1$.

i) Model the process $y(n)$ as an AR(2) process. What is the conditional variance $\mathrm{Var}(y(n+1)|y(n), y(n-1))$ of the modeled process? (3p)

ii) You want to predict the next value $y(n+1)$ of the process $y(n)$, but you only have available distorted observations $x(n) = y(n) + v(n)$, where the complex-valued WSS process $v(n)$ is uncorrelated with the process $y(n)$ and you know its autocorrelations $r_v(0) = 4$, $r_v(1) = 2$, $r_v(2) = 1$. The processes $y(n)$ and $v(n)$ are jointly WSS. Solve a FIR Wiener filter whose task is to predict the value $y(n+1)$ from the values $x(n)$ and $x(n-1)$. What is the mean squared prediction error of your Wiener filter? Compare it to the conditional variance in part i). (3p)

3. (max 6p)

Answer the following propositions either "true" or "false": you may also leave any of them unanswered. A correct answer gives 1 point, a wrong answer -1 points, and a missing answer zero points. However, the total number of points you receive from this problem cannot become negative; the total number of points is at least zero. No need to justify your answers.

a) The ARMA(2,1) process $x(n) = -(1/6)x(n-1) + (1/6)x(n-2) + v(n) + 4v(n-1)$, where $v(n)$ is normally distributed white noise with variance 1, is wide sense stationary.

b) If we estimate a $M \times M$ autocorrelation matrix from N observations, the covariance method computes each of its estimates from at least equally many observations as the autocorrelation method.

c) Let $v(n)$ be white noise. Then the process $x(n) = v(n) + 0.5v(n-1)$ is also white noise.

d) In Pisarenko's method one must know more values of the autocorrelation than the number of sinusoids in the process; in the MUSIC method one must know at least equally many values of the autocorrelation as in Pisarenko's method.

e) The autocorrelation $r_y(n)$ of the output is the convolution of the autocorrelation $r_x(n)$ of the input and the unit sample response $h(n)$.

f) If $x(n) = d(n) + v(n)$, $x(n)$ and the desired signal $d(n)$ are jointly WSS, and $d(n)$ and the white noise $v(n)$ are uncorrelated, then based on these facts it is possible to give an upper limit to the frequency response of an IIR Wiener filter.

4. (max 6p)

You have measured the following three observations from the real-valued zero-mean WSS process $x(n)$: $x(0) = 2$, $x(1) = 1$, $x(2) = 1$. You know that the process should have a lot of power in one frequency, which is either $\pi/3$ or $\pi/2$.

i) Estimate a 3×3 autocorrelation matrix for the process so that the result is positive semidefinite. You do not need to prove the positive semidefiniteness. (1.5p)

ii) Estimate the value of the power spectrum for the two above-mentioned frequencies using the periodogram. Based on the estimates, which frequency is more likely to be correct? (1.5p)

iii) Estimate the value of the power spectrum for the two above-mentioned frequencies or compute the value of the pseudospectrum for the frequencies, using a parametric method of your choice. Based on the results, which frequency is more likely to be correct? You can use a smaller 2×2 autocorrelation matrix if the computation in your chosen method is otherwise too difficult. (3p)

Possibly useful information:

$\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, $\cos(\frac{\pi}{3}) = \frac{1}{2}$, $\sin(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$, $\cos(\frac{2\pi}{3}) = -\frac{1}{2}$.

T-61.3040 Statistical Modeling of Signals

Answers to Final Exam 31.8.2007

In the exam you are allowed to have a calculator (non-programmable or memory emptied) and basic mathematical tables (no tables containing material directly associated with the course)

1. (max 6p)

Explain *briefly* the following topics without unnecessary detail:

i) Wide-sense stationarity (WSS) (2p)
Answer: A random process is called wide-sense stationary if it satisfies three conditions: the process has a *constant mean* (0.5p), *finite variance* (0.5p) and the *autocorrelation between two points only depends on their time difference* (1p).

ii) Ergodicity (2p)
Answer (several possible ways to get 2 points): A wide-sense stationary random process is called *ergodic in the mean* if the *sample mean* (over time) converges to the *ensemble mean* (over different realizations of the random process) (1.5p). A WSS process is called *autocorrelation ergodic* if autocorrelation estimated from the sample mean converges to the true value in the mean-square sense (1p). *Mean ergodic theorems:* a WSS process is ergodic in the mean if the mean of the autocovariance sequence converges to zero (1p), or if the first autocovariance is finite and the autocovariances go to zero (1p).

iii) Wiener filter (2p)
Answer (several possible ways to get 2 points): A Wiener filter estimates a WSS signal $d(n)$ from observations $x(n) = d(n) + v(n)$ having WSS noise, by minimum mean-square error estimation (1.5p). A FIR Wiener filter can be solved by the Wiener-Hopf equations $\mathbf{R}_x\mathbf{w} = \mathbf{r}_{dx}$. Further simplification is possible with more knowledge about the signal and noise statistics (0.5p). Wiener filtering can be used for linear prediction (in noise, known autocorrelation) from current and previous values, or for noise cancellation (noise autocorrelation from secondary sensor) (1p). Adaptive Wiener filtering is also possible; it converges to the Wiener-Hopf solution under certain conditions (1p).

2. (max 6p)

You have observed the following values of the real-valued process $x(n)$: $x(0) = 1$, $x(1) = 2$, $x(2) = 2$.

i) Estimate an autocorrelation matrix of size 3×3 so that the result is positive semidefinite. Show that the result is positive semidefinite. (2p)
Answer: We will use the autocorrelation method: $\hat{r}_x(k) = (1/(N+1)) \sum_{n=k}^{N} x(n)x(n-k)$. Thus the result is:

$$r_x(0) = (1/3)(1 \cdot 1 + 2 \cdot 2 + 2 \cdot 2) = 9/3$$

$$r_x(1) = (1/3)(1 \cdot 0 + 2 \cdot 1 + 2 \cdot 2) = 6/3$$

$$r_x(2) = (1/3)(1 \cdot 0 + 2 \cdot 0 + 2 \cdot 1) = 2/3$$

and the covariance matrix is

$$\hat{\mathbf{R}}_x = \frac{1}{3}\begin{bmatrix} 9 & 6 & 2 \\ 6 & 9 & 6 \\ 2 & 6 & 9 \end{bmatrix}.$$

Positive semidefiniteness: writing the above equations in a special matrix form (see slides page 263) we have

$$\hat{\mathbf{R}}_x = (1/3)\begin{bmatrix} x(0) & 0 & 0 \\ x(1) & x(0) & 0 \\ x(2) & x(1) & x(0) \\ 0 & x(2) & x(1) \\ 0 & 0 & x(2) \end{bmatrix}^T \begin{bmatrix} x(0) & 0 & 0 \\ x(1) & x(0) & 0 \\ x(2) & x(1) & x(0) \\ 0 & x(2) & x(1) \\ 0 & 0 & x(2) \end{bmatrix}$$

and therefore $\mathbf{s}^T\hat{\mathbf{R}}_x\mathbf{s} = \mathbf{s}^T\frac{1}{3}(\mathbf{H}^T\mathbf{H})\mathbf{s} = \frac{1}{3}||\mathbf{H}\mathbf{s}||^2 \geq 0$ where $\mathbf{H}$ is the matrix in the previous equation. This satisfies the definition of positive semidefiniteness. (The precise values of $x(i)$ do not matter here; the result is always a positive semidefinite matrix.)

ii) Model the process $x(n)$ as an $AR(2)$ process. (2p)
Answer: An AR(p) process has the form $x(n) = -\sum_{k=1}^{p} a(k)x(n-k) + v(n)$. When $p = 2$ this becomes $x(n) = -a(1)x(n-1) - a(2)x(n-2) + v(n)$. The weights $a(1)$ and $a(2)$ can be solved from the autocorrelations, using the Yule-Walker equations:

$$\begin{bmatrix} r_x(0) & r_x^*(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = -\begin{bmatrix} r_x(1) \\ r_x(2) \end{bmatrix}.$$

Inserting the estimated values of the autocorrelations, the equations become:

$$\frac{1}{3}\begin{bmatrix} 9 & 6 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

The parameters can be solved by multiplying from the left by the inverse of the matrix:

$$\begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 6 & 9 \end{bmatrix}^{-1} \begin{bmatrix} -6 \\ -2 \end{bmatrix} = \frac{1}{9\cdot 9 - 6\cdot 6}\begin{bmatrix} 9 & -6 \\ -6 & 9 \end{bmatrix}\begin{bmatrix} -6 \\ -2 \end{bmatrix} = \begin{bmatrix} -14/15 \\ 2/5 \end{bmatrix}.$$

iii) What are the variance $\mathrm{Var}(x(n))$ and conditional variance $\mathrm{Var}(x(3)|x(2), x(1), x(0))$ of the modeled process? (2p)
Answer: For the variance we have $\mathrm{Var}(x(n)) = E[x(n)^2] - (E[x(n)])^2$. The latter term is zero because the process is zero-mean: taking expectations of the process equation we have $E[x(n)] = -a(1)E[x(n)] - a(2)E[x(n)] + 0 = (8/15)E[x(n)] \Rightarrow E[x(n)] = 0$. The former term is just $r_x(0)$, so the result is $\mathrm{Var}(x(n)) = r_x(0) = 9/3 = 3$.

For the conditional variance we have $Var(x(3)|x(2), x(1), x(0)) = Var(x(3)|x(2), x(1)) = var(v(3)) = |b(0)|^2$ since the other terms are constant given $x(2)$ and $x(1)$. For $|b(0)|^2$ we have the equation $|b(0)|^2 = r_x(0) + a(1)r_x(-1) + a(2)r_x(-2) = 3 - \frac{14}{15}\cdot 2 + \frac{2}{5}\cdot\frac{2}{3} = \frac{45-28+4}{15} = \frac{21}{15} = \frac{7}{5} = 1.4$.

3. (max 6p)
Answer the following propositions either "true" or "false": you may also leave any of them unanswered. A correct answer gives 1 point, a wrong answer -1 points, and a missing answer zero points. No need to justify your answers.

a) The Cramer-Rao lower bound is the smallest variance that an unbiased estimator can achieve.
Answer: true. See lecture 2.

b) For both the power spectrum and the pseudospectrum, the integral over some frequency band tells the power of the process on that band.
Answer: false. The pseudospectrum does not contain any information about the power in the complex exponentials.

c) The autocorrelation $r_y(n)$ of the output is the convolution of the autocorrelation $r_x(n)$ of the input and the unit sample response $h(n)$.
Answer: false. Instead, there is a double convolution: $r_y(n) = h(n) * h^*(-n) * r_x(n)$, see lecture 5.

d) An MA(q) process has infinitely many nonzero autocorrelations $r_x(k)$.
Answer: false. An MA(q) process has at most $q + 1$ nonzero autocorrelations, see lecture 4.

e) The resolution of the periodogram improves as the number of data grows.
Answer: true. The resolution depends on the width of the Bartlett window and improves (becomes smaller) as $1/N$ with the number of samples N.

f) The LMS algorithm always converges if the step size μ satisfies the condition $0 < \mu < 2$.
Answer: false. It converges when $x(n)$ and $d(n)$ are together wide-sense stationary and $0 < \mu < 2/\lambda_{max}$ where λ_{max} is the largest eigenvalue of the correlation matrix $\mathbf{R}_x$ of $x(n)$.

4. (max 6p)
The process $x(n) = A\exp(jn\omega) + v(n)$ consists of one complex sinusoid in white noise. If the correlation matrix is

$$R_x = \begin{bmatrix} 3 & 2(1-j) \\ 2(1+j) & 3 \end{bmatrix}$$

then

i) what is the variance of the noise? (2p)

ii) what is the power $P = |A|^2$? (2p)
Answer: from the process equation we can derive the equations for the autocorrelations:

$$r_x(0) = E[(A\exp(jn\omega) + v(n))(A\exp(jn\omega) + v(n))^*] = \sigma^2 + E[AA^*\exp(jn\omega - jn\omega)] = \sigma^2 + |A|^2 ,$$

$$\begin{aligned} r_x(1) = E[(A\exp(j(n+1)\omega) + v(n+1))(A\exp(jn\omega) + v(n))^*] &= 0 + E[AA^*\exp(j(n+1)\omega - jn\omega)] \\ &= |A|^2 E[\exp(j\omega)] = |A|^2\exp(j\omega) . \end{aligned}$$

Because a complex exponential has radius 1, we have $|A|^2 = |r_x(1)| = |2(1+j)| = \sqrt{2^2+2^2} = 2\sqrt{2}$ and therefore $\sigma^2 = r_x(0) - |A|^2 = 3 - 2\sqrt{2}$.

iii) what is the frequency ω of the signal? (2p) We have

$$r_x(1) + r_x(1)^* = |A|^2\exp(j\omega) + |A|^2\exp(-j\omega) = 2|A|^2\cos(\omega) = 2(1+j) + 2(1-j) = 4$$

and therefore

$$\cos(j\omega) = \frac{4}{2|A|^2} = \frac{4}{2\cdot 2\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \omega = \pm\frac{\pi}{4} .$$

We can check the correct sign of ω from the imaginary part of $r_x(1)$: $|A|^2 exp(+j\pi/4) = 2\sqrt{2}(\cos(\pi/4) + j\sin(\pi/4)) = 2\sqrt{2}(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})$ which is correct and so $\pi/4$ is the correct frequency.