1st mid-term exam 22.2.2011 at 16–19. L4

No calculators, no tables.

1. The function f is 2π-periodic and $f(x) = e^{-ix/2}$ for $-\pi \le x \le \pi$. Calculate the complex Fourier coefficients c_n of f.

2. a) Give a short explanation how to derive the formula for the complex Fourier coefficients c_n of a 2π-periodic function, using the orthogonality of the functions $e_n(x) = e^{inx}$ with respect to the inner product of L^2.

   b) Derive the formula $\widehat{u'(x)}(\xi) = i\xi\hat{u}(\xi)$, when $u\colon \mathbf{R} \to \mathbf{R}$ and u' are smooth and L^1-integrable.

3. a) Show that the solution of $u'' + 2u' + u = f(x)$ has a Fourier representation of the form $\hat{u}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$ for $f \in L^1(\mathbf{R})$ and

$$\hat{g}(\xi) = \frac{1}{(1+i\xi)^2}.$$

   b) Determine $g(x)$ in some way and give a convolution formula for $u = u(x)$.

4. Let us consider a damped wave equation of the form

$$\begin{cases} u_{tt} + 2\alpha u_t = c^2 u_{xx}, & 0 < x < \pi,\ t > 0 \\ u(0,t) = 0, & t \ge 0 \\ u_x(\pi,t) = 0, & t \ge 0 \text{ (huomaa derivaatta!)} \\ u(x,0) = f(x), & 0 \le x \le \pi \\ u_t(x,0) = 0, & 0 \le x \le \pi, \end{cases}$$

   where $0 < \alpha < c/2$. Using separation of variables, derive a (formal) solution formula

$$u(x,t) = \sum_{n=0}^{\infty} e^{-\alpha t}(A_n \cos(\beta_n t) + B_n \sin(\beta_n t)) \sin(\lambda_n x),$$

   determine the parameters λ_n, β_n and the coefficients A_n, B_n in terms of the function f.

Some formulas:

- $\widehat{f * g} = \hat{f}\hat{g}$, $\widehat{fg} = (2\pi)^{-n}\hat{f} * \hat{g}$.
- $\hat{h}(\xi) = 1/(a + i\xi)$, where $h(x) = H(x)e^{-ax}$, $a > 0$ and

$$H(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0. \end{cases}$$