Aalto University, School of Science

Mat-1.3651 Matrix computations **Eirola/Murtola**

Exam May 16, 2011

No calculators or texts are allowed. Time for the exam is three hours.

1. Let $\boldsymbol{S} \in \mathbb{C}^{n\times n}$ be skew-Hermitian, i.e., $\boldsymbol{S}^* = -\boldsymbol{S}$.
   a) Show that the eigenvalues of $\boldsymbol{S}$ are purely imaginary.
   b) Show that the *Cayley transform*: $\boldsymbol{Q} = (\boldsymbol{I} - \boldsymbol{S})^{-1}(\boldsymbol{I} + \boldsymbol{S})$ is unitary.

2. Assume $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, $m \geq n$ has full rank. Consider the block system of equations
$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{A} \\ \boldsymbol{A}^* & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ 0 \end{bmatrix} .$$
   Show that this system has a unique solution $\left[\begin{smallmatrix} \boldsymbol{r} \\ \boldsymbol{x} \end{smallmatrix}\right]$, and the vectors $\boldsymbol{x}$ and $\boldsymbol{r}$ are the solution and the residual, respectively, of the least squares problem: $\min_{\boldsymbol{x}} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2$.

3. Assume $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ are symmetric and positive definite. Show that the 2-norm condition number satisfies
$$\kappa_2(\boldsymbol{A} + \boldsymbol{B}) \leq (\max \Lambda(\boldsymbol{A}) + \max \Lambda(\boldsymbol{B}))/(\min \Lambda(\boldsymbol{A}) + \min \Lambda(\boldsymbol{B})) .$$

4. Let $\boldsymbol{A} = \left[\begin{smallmatrix} 0 & \boldsymbol{I} \\ \boldsymbol{I} & 0 \end{smallmatrix}\right] \in \mathbb{R}^{2n\times 2n}$. Consider solving $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ with the conjugate gradient method. Show that it will find the solution in two steps.
   Hint: What does the method minimize?