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Mat-1.3650 Finite element method I

Final exam, 23.5.2011

1. Consider the following affine transformation from the reference element $\hat{K}$ to an element K:
$$\boldsymbol{x} = \boldsymbol{A}_K \hat{\boldsymbol{x}} + \boldsymbol{b}_K$$
and let $u(\boldsymbol{x}) = u(\boldsymbol{A}_K \hat{\boldsymbol{x}} + \boldsymbol{b}_K)$.
   - Transform the gradient ∇u to the reference element.
   - Transform the integral $\int_K \nabla u \cdot \nabla v d\boldsymbol{x}$ to the reference element.
   - Compute the above integral using a simple quadrature rule. Use the midpoint of the reference element as the quadrature point and $\frac{1}{2}$ as the integration weight.

2. Let $a(u,v) = (\nabla u, \nabla v)$ and $l(v) = \langle f, v\rangle$, and Ω is a domain with a smooth boundary.
   - Show that the energy minimization problem
$$\min_{v \in H_0^1(\Omega)} \frac{1}{2} a(v,v) - l(v)$$
is satisfied for $u \in H_0^1(\Omega)$ *if and only if* for every $v \in H_0^1(\Omega)$ it holds
$$a(u,v) = l(v).$$
   - Show that a solution exists and is unique using the Lax-Milgram theorem.

3. Using the transformation formulae
$$|\hat{v}|_{m,\hat{K}} \leq C_1 \|\boldsymbol{A}_K\| |\det \boldsymbol{A}_K|^{-1/2} |v|_{m,K}$$
$$|v|_{m,K} \leq C_2 \|\boldsymbol{A}_K^{-1}\| |\det \boldsymbol{A}_K^{-1}|^{-1/2} |\hat{v}|_{m,\hat{K}}$$
in both directions, and employing the Bramble-Hilbert lemma
$$\|\hat{u} - I\hat{u}\|_{1,\hat{K}} \leq C(\hat{K}) |\hat{u}|_{2,\hat{K}},$$
in which Iu is the linear interpolant of u prove the estimate
$$\|u - Iu\|_{1,K} \leq Ch|u|_{2,K}$$
with $h = \max_{K \in \mathcal{K}_h} h_K$.
Hint: Assume a uniform, shape-regular mesh and use the relation $\|\boldsymbol{A}_K\| \sim h_K$ and $\|\boldsymbol{A}_K^{-1}\| \sim h_K^{-1}$.

4. Let u_h be the solution of the weak problem

$$(\nabla u_h, \nabla v) = (f, v), \quad \forall v \in V_h,$$

and u the corresponding exact solution.

- Show that
$$\|u - u_h\|_1 \leq C \inf_{v \in V_h} \|u - v\|_1.$$

- Combine the previous results with Problem 3 to show the error estimate
$$\|u - u_h\|_1 \leq Ch|u|_2.$$