Quantum Physics TFY-3.4323
Second midterm examination
18 May 2011

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1. Explain briefly the following concepts. (You may use equations if you find it convenient, but it is not necessary.)

   a) Markov approximation.

   b) Quantum gas.

   c) Quantum phase transition.

   d) Cooper pair.

2. Consider non-interacting bosons in a cube of volume $V = L^3$, in which L is the length of the cube. Show that there is Bose-Einstein condensation for certain particle numbers and temperatures.

   Guidelines: The bosons are described by the Hamiltonian

$$H = \int d\mathbf{x}\, \psi^\dagger(\mathbf{x}) \left(-\frac{\hbar^2 \nabla^2}{2m}\right) \psi(\mathbf{x}).$$

   Assuming a periodic boundary condition, the single particle eigenstates are

$$\varphi_\mathbf{n}(\mathbf{x}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}_\mathbf{n}\cdot\mathbf{x}}$$

   with the wave-vector $\mathbf{k}_\mathbf{n} = \frac{2\pi}{L}(n_x, n_y, n_z)$, $n_i = 0, \pm 1, \pm 2, \ldots$ At an inverse temperature $\beta = \frac{1}{k_B T}$ and chemical potential μ the expected number of particles in state $\varphi_\mathbf{n}$ is

$$n_\mathbf{n} = \langle \hat{a}_\mathbf{n}^\dagger \hat{a}_\mathbf{n} \rangle = \mathrm{Tr}\{\hat{\rho}\hat{a}_\mathbf{n}^\dagger \hat{a}_\mathbf{n}\} = \frac{ze^{-\beta E_\mathbf{n}}}{1 - ze^{-\beta E_\mathbf{n}}},$$

   with the notation $z = e^{\beta\mu}$. The total number of particles in the system is

$$N = \sum_\mathbf{n} n_\mathbf{n} = \sum_\mathbf{n} \frac{ze^{-\beta E_\mathbf{n}}}{1 - ze^{-\beta E_\mathbf{n}}}.$$

   You may find it useful to estimate a discrete sum with an integral, to employ spherical coordinates and to use the following function

$$g_{\frac{3}{2}}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2 \frac{ze^{-x^2}}{1 - ze^{-x^2}}, \qquad g_{\frac{3}{2}}(1) = 2.612\ldots$$

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3. a) What does quantum teleportation mean?

   b) Why does the quantum teleportation protocol not contradict the special theory of relativity?

   c) What is the fate of the particles left behind (in the sender's possession) after teleportation?

   d) Show that the operation $U_\psi = 2\,|\psi\rangle\,\langle\psi| - I$, which appears in the Grover iteration can be regarded as an *inversion about the mean*. Remember that here $|\psi\rangle = \frac{1}{\sqrt{N}} \sum\limits_{x=0}^{N-1} |x\rangle$. In other words show that U_ψ applied to an arbitrary state $\sum\limits_{k=0}^{N-1} \alpha_k\,|k\rangle$ produces $\sum\limits_{k=0}^{N-1} (2\bar{\alpha} - \alpha_k)\,|k\rangle$, where $\bar{\alpha} = \frac{1}{N} \sum\limits_{k=0}^{N-1} \alpha_k$.

4. To describe e.g. bosonic atoms in an optical lattice one may use the Bose-Hubbard Hamiltonian

$$H = -J \sum_{<i,j>} a_i^\dagger a_j + \frac{U}{2} \sum_i a_i^\dagger a_i^\dagger a_i a_i$$

together with the Gutzwiller mean-field ansatz

$$|\Psi_{MF}\rangle = \prod_{i=1}^{M} \left[\sum_{n_i=0}^{\infty} f_{n_i}^{(i)} |n_i\rangle \right].$$

   a) Calculate the following quantities:

$$\langle \Psi_{MF} | H | \Psi_{MF} \rangle,$$
$$\langle \Psi_{MF} | \hat{n}_i | \Psi_{MF} \rangle,$$
$$\langle \Psi_{MF} | a_i | \Psi_{MF} \rangle,$$
$$\sigma_i^2 = \frac{\langle \Psi_{MF} | \hat{n}_i^2 | \Psi_{MF} \rangle - \langle \Psi_{MF} | \hat{n}_i | \Psi_{MF} \rangle^2}{\langle \Psi_{MF} | \hat{n}_i | \Psi_{MF} \rangle}.$$

   (We have the notation $\hat{n}_i = a_i^\dagger a_i$.)

   b) Explain briefly how you can distinguish between a Mott insulator and a superfluid phase based on the quantities above.