S-88.4200 Statistical Signal Processing.
Final Exam May 23, 2011

1. Define or explain briefly the following concepts:

   (a) Bias and variance of an estimator

   (b) Consistency of estimates

   (c) Efficiency of an estimator

   (d) Fisher Information matrix

   (e) Maximum Likelihood estimator

   (f) Bayes Risk

2. Unbiased Estimator
   Consider the i.i.d. data $\{x(0), x(2), \ldots, x(N-1)\}$, where each sample is distributed as $\mathcal{U}(0, a)$ (uniform distributed between 0 and a).

   (a) Find an unbiased estimator for a. The range of a is $0 < a < \infty$.

   (b) Show that the estimator derived under (a) is indeed an unbiased estimator for a.

3. Minimum Variance Unbiased Estimator

   (a) Explain briefly the concept Minimum Variance Unbiased Estimator.

   (b) Describe how Minimum Variance Unbiased Estimators are derived using concepts of sufficient statistics and completeness. Explain what is done in each step.

4. Suppose that $y_1, y_2, \ldots, y_N$ are independent samples of a Gaussian random variable with distribution $\mathcal{N}(\theta, \sigma_v^2)$ where θ is a random variable with a priori distribution $\mathcal{N}(a, \sigma_\theta^2)$. Assume that σ_v^2, σ_θ^2, and a are known.

   a) Find the Bayesian maximum a posteriori estimator $\hat{\theta}_{MAP}$ of θ.

   b) Find the Bayesian estimator $\hat{\theta}_q$ of θ that minimizes the quadratic risk function.

5. A random parameter θ is uniformly distributed between $(-10, 10)$. Find the Mean Square estimate of θ when we have made the 2 observations $y(1) = -0.5$ and $y(2) = 1.0$. The statistically independent measurements $y(i)$ obey the distribution

$$f(y(i)|\theta) = \begin{cases} y(i) - \theta + 1, & \theta - 1 \leq y(i) \leq \theta \\ -y(i) + \theta + 1, & \theta < y(i) \leq \theta + 1 \\ 0, & otherwise \end{cases}$$

Possibly useful formulas

The Bayes' rule:

$$f(\theta|\mathbf{y}) = \frac{f(\mathbf{y}|\theta)f(\theta)}{f(\mathbf{y})}$$

Binomial distribution

$$P_X(k) = \begin{cases} nkp^k(1-p)^{n-k} & k = 0, 1, 2, \dots, n \\ 0 & otherwise. \end{cases}$$

and $0 < \theta < 1$

Normal Distribution

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$$

Multivariate Gaussian with mean vector μ and covariance matrix $\mathbf{R}$

$$f(\mathbf{y}) = \frac{1}{\sqrt{(2\pi)^N|R|}} exp \frac{-1}{2}[\mathbf{y} - \mu]\mathbf{R}^{-1}[\mathbf{y} - \mu]^{\mathbf{T}}$$

Conditional Mean when $\mathbf{x}$ and $\mathbf{y}$ are jointly Gaussian distributed

$$\mathrm{E}\{\mathrm{y}|\mathrm{x}\} = \mathrm{E}\{\mathrm{y}\} + \mathbf{R}_{\mathrm{yx}}\mathbf{R}_{\mathrm{xx}}^{-1}(\mathrm{x} - \mathrm{E}\{\mathrm{x}\}),$$

with

$$\mathbf{R}_{\mathrm{xx}} = \mathrm{E}\left\{(\mathrm{x} - \mathrm{E}\{\mathrm{x}\})(\mathrm{x} - \mathrm{E}\{\mathrm{x}\})^{\mathrm{T}}\right\},$$

and

$$\mathbf{R}_{\mathrm{yx}} = \mathrm{E}\left\{(\mathrm{y} - \mathrm{E}\{\mathrm{y}\})(\mathrm{x} - \mathrm{E}\{\mathrm{x}\})^{\mathrm{T}}\right\}.$$