S-88.4101 Sensor Array Signal Processing.
Final Exam Dec. 15, 9:00-12:00, 2011, Room A102 (S1)

1. Define or explain briefly the following terms and concepts: **(9P)**
   (a) Spatial Filtering (1P)
   (b) Beamformer (1P)
   (c) Manifold Separation (2P)
   (d) MVDR Beamformer (2P)
   (e) Conditional and Unconditional Data Model (3P)

2. Sampling Interval **(3P)**
   You are given the task of designing an ideal uniform linear array composed of nine omnidirectional sensors. What is the maximum inter-element spacing (and consequently the maximum array aperture) the sensor array should have, such that the directions of sources in the angular sector $\frac{\pi}{6} \leq \varphi \leq \frac{3\pi}{4}$ can be estimated unambiguously in the absence of noise. The co-elevation of the sources is known to be $\vartheta = \frac{\pi}{2}$. The broadside direction is $\varphi = \frac{\pi}{2}$, i.e. the uniform linear array is parallel to the x-axis of the coordinate system.

3. Direction Vector of an UCA **(3P)**
   Assume a uniform circular array with omnidirectional sensors. The uniform circular array lays in the x-y-plane. It has M elements and radius r. Derive the direction vector $\mathbf{a}(\varphi, \vartheta)$ for a far-field narrow-band source at azimuth φ and co-elevation ϑ.

4. Signal Subspace Eigenvalues **(5P)**
   Assume a covariance matrix $\mathbf{R}_{xx} \in \mathbb{C}^{M \times M}$ has structure

$$\mathbf{R}_{xx} = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \sigma^2\mathbf{I} \in \mathbb{C}^{M \times M},$$

   where $\mathbf{A}$ has full column rank. The covariance matrix $\mathbf{R}_{xx}$ has eigenvalue decomposition

$$\mathbf{R}_{xx} = \left[\begin{array}{cc} \mathbf{U}_S & \mathbf{U}_N \end{array}\right] \left[\begin{array}{cc} \mathbf{\Lambda}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_n \end{array}\right] \left[\begin{array}{cc} \mathbf{U}_S & \mathbf{U}_N \end{array}\right]^H.$$

   Show that

$$\mathbf{\Lambda}_s^{-1} = \sigma^{-2}\mathbf{I} - \sigma^{-4}\mathbf{U}_s^H\mathbf{A}\left(\sigma^{-2}\mathbf{R}_{ss}\mathbf{A}^H\mathbf{A} + \mathbf{I}\right)^{-1}\mathbf{R}_{ss}\mathbf{A}^H\mathbf{U}_s$$

5. MUSIC Pseudo-Spectrum **(5P)**
Describe the MUSIC algorithm leading to the MUSIC Pseudo-Spectrum, assuming the array-manifold $\mathbf{a}(\varphi)$ is known and only a function of the azimuth φ.

6. Solve *one* of the following problems: **(5P)**

a) System for DoA and Signal Estimation
You are given the task to design a system for DoA and signal estimation, with the following specification

- An uniform linear array is used to acquire the data, the inter-element spacing is $d' = 0.45\lambda$, the number of sensors is M.
- The structure of the data is

$$\mathbf{x}(n) = \mathbf{A}(\boldsymbol{\varphi})\mathbf{s}(n) + \mathbf{w}(n).$$

- The observation noise is i.i.d. Gaussian.
- There are $K < \frac{M}{2}$ signals, and it is known that they are non-coherent.
- The number of samples N is significantly larger than the number of sensors, i.e. $N >> M$.
- The system should rely as much as possible on search-free algorithms.

(a) Design a system to estimate the number of sources, their direction of arrival φ and the signal samples $\mathbf{s}(n)$.

(b) Describe your signal processing system using, e.g. a block diagram on a *high level*.

(c) Write what quantities are passed on between your processing blocks and where the estimates $\hat{\mathbf{s}}(n)$, $\hat{\boldsymbol{\varphi}}$, and $\hat{K}$ are determined.

(d) Motivate shortly your choices for the algorithms in your processing blocks.

b) Derivative of Projection Matrix
Let $\mathbf{P} = \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H = \mathbf{A}\mathbf{A}^+$, and $\mathbf{P}^\perp = \mathbf{I} - \mathbf{P}$. Furthermore, let φ be a real scalar and $\mathbf{A} = \mathbf{A}(\varphi)$. Let

$$\mathbf{A}_\varphi = \frac{\partial \mathbf{A}}{\partial \varphi} = \frac{\partial \mathbf{A}(\varphi)}{\partial \varphi}$$

be element-wise differentiation of the matrix $\mathbf{A}$ to φ. Show that the derivative of $\mathbf{P}^\perp$ to φ is given by

$$\mathbf{P}^\perp_\varphi = -\mathbf{P}^\perp\mathbf{A}_\varphi\mathbf{A}^+ - \left(\mathbf{P}^\perp\mathbf{A}_\varphi\mathbf{A}^+\right)^H .$$

$\mathbf{C}^+$ denotes the pseudo-inverse of the matrix $\mathbf{C}$, $\mathbf{C}^H$ denotes its hermitian-transpose.

Possibly useful formulas

- Using the wave number one can write in compact form for the vector sample of the wavefield $\mathbf{x}(t)$, having M elements:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_M(t) \end{bmatrix} = \begin{bmatrix} e^{-j\mathbf{k}_T^T\mathbf{p}_1} \\ \vdots \\ e^{-j\mathbf{k}_T^T\mathbf{p}_M} \end{bmatrix} x_0(t) = \mathbf{a}(\mathbf{k})x_0(t).$$

- The wave number $\mathbf{k}$ is defined as

$$\mathbf{k} = -\frac{2\pi}{\lambda_c}\mathbf{u}_T = -\frac{\omega}{c_0}\mathbf{u}_T$$

- The vector $\mathbf{u}_T$ can be expressed using azimuth angle φ_T and co-elevation angle ϑ_T by

$$\mathbf{u}_T = \begin{bmatrix} \sin(\vartheta_T)\cos(\varphi_T) \\ \sin(\vartheta_T)\sin(\varphi_T) \\ \cos(\vartheta_T) \end{bmatrix}$$

- Some trigonometric identities

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \sin(y)\cos(x)$$
$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

- For uniform linear sampling along the y-axis the following inequality has to hold

$$\left|\frac{d'}{\lambda_c}\sin(\varphi_T)\right| < \frac{1}{2}, \forall\varphi_T$$

- $\partial\mathbf{X}^{-1} = -\mathbf{X}^{-1}\partial\mathbf{X}\mathbf{X}^{-1}$
- $\partial\mathbf{X}^H = \partial(\mathbf{X}^H) = (\partial\mathbf{X})^H$
- $\partial(\mathbf{X}\mathbf{Y}) = \partial\mathbf{X}\mathbf{Y} + \mathbf{X}\partial\mathbf{Y}$
- Matrix Inversion Lemma

$$(\mathbf{A}+\mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}\mathbf{A}^{-1}\mathbf{B}+\mathbf{C}^{-1}\right)^{-1}\mathbf{D}\mathbf{A}^{-1}$$