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T-79.5205 Combinatorics (5 cr)
Exam Wed 24 Aug 2011, 9–12 a.m.

Write down on each answer sheet:
- Your name, degree programme, and student number
- The text: "T-79.5205 Combinatorics 24.8.2011"
- The total number of answer sheets you are submitting for grading

Note: You can write down your answers in either Finnish, Swedish, or English.

1. Give a closed-form solution or a counting recurrence as a function of n and k.

   (a) Determine the number of distinct nonnegative integer solutions $(x_1, x_2, \ldots, x_n)$ to the equation $x_1 + x_2 + \ldots + x_n = k$.

   (b) You have k one-euro coins and n brown envelopes. Determine the number of different ways to place the euro coins into the envelopes so that each envelope contains at least one coin.

   (c) You have k one-euro coins. Determine the number of different ways to distribute the coins to your n friends so that each friend receives at least one coin.

2. Using the principle of inclusion and exclusion, derive a formula for the number of solutions $(x_1, x_2, \ldots, x_n) \in \{0, 1, \ldots, B-1\}^n$ to the equation $x_1 + x_2 + \ldots + x_n = k$.

   Hint: If $x_i \geq B$, we have $x_i - B \geq 0$. Use Problem 1(a). Does your formula simplify to $\binom{n}{k}$ for $B = 2$?

3. Partially ordered sets.

   (a) Let $[n] = \{1, 2, \ldots, n\}$ and denote by Π_n the set of all set partitions of $[n]$. For set partitions $\sigma = \{S_1, S_2, \ldots, S_p\}$ and $\tau = \{T_1, T_2, \ldots, T_q\}$ of $[n]$, define $\sigma \leq \tau$ if and only if for every $i = 1, 2, \ldots, p$ there exists a $j = 1, 2, \ldots, q$ with $S_i \subseteq T_j$. Show that Π_n is partially ordered by $\leq$.

   (b) A chain in a partially ordered set is *maximal* if the chain is not a proper subset of a larger chain. For $n = 6$, give an example of a maximal chain in $(\Pi_n, \leq)$.

   (c) Derive an expression for the number of maximal chains in $(\Pi_n, \leq)$.

4. Graphs and symmetry.

   (a) Give an example of a graph with at least two vertices whose only automorphism is the identity permutation. Carefully justify why this is the case.

   Hint: You will need at least six vertices.

   (b) For each integer $n \geq 2$, give an example of a graph with exactly n distinct automorphisms.

   Hint: For $n \geq 3$, use part (a) as a building block.

Grading: Each problem 12p, total 48p.