Aalto University School of Science
Department of Information and Computer Science
Pekka Orponen (tel. 25246)

T-79.5203 Graph Theory (5 cr)
Exam Tue 24 May 2011, 1-4 p.m.

Write down on each answer sheet:
- Your name, degree programme, and student number
- The text: "T-79.5203 Graph Theory 24.5.2011"
- The total number of answer sheets you are submitting for grading

Note: You can write down your answers in either Finnish, Swedish, or English.

1. (a) Draw all the nonisomorphic (undirected, simple, loopless) graphs on two and four vertices. (*All* means unconnected as well as connected ones.)
   (b) Study the attached webpage on graph products and draw the graphs $P_3 \times P_3$ and $P_3 \bullet P_3$.
   (c) Give an example of a graph with four vertices that *cannot* be obtained as any kind of product of two graphs with two vertices. (In the sense described on the webpage, and considering all the 256 types of products.) Prove that your graph has the desired property.

2. Five students throw a frisbee. The students wonder about the length of the longest sequence of throws such that each ordered sender-receiver pair (that is, $A \to B$ differs from $B \to A$) occurs at most once.
   (a) Please help the students to determine this length, and also present one possible sequence of maximum length.
   (b) Can you say something about the same problem with $n = 2011$ students? (Do *not* attempt to list such sequences.)
   (c) For an arbitrary n, determine the *shortest* length of a *maximal* sequence, that is, a sequence that cannot be continued without violating the above mentioned condition.

3. Suppose a Computer Science degree programme consists of n courses, all of them mandatory. The prerequisite (di)graph G for the programme has a vertex for each course, and a directed edge from course u to course v if and only if u is a prerequisite of v. (We shall assume that the graph G contains no cycles.) Give a linear-time algorithm that takes as input the graph G and determines the minimum number of semesters necessary to complete the programme, assuming that a student can take any number of courses in one semester. Justify the correctness and complexity of your algorithm.

4. A contingent of N families goes out to dinner together to a restaurant with M tables. To increase their social interaction, the families wish to be seated so that no two members of the same family are at the same table. Give an efficient algorithm for computing such a seating assignment, or determining that none is possible. Each family i has n_i members, and each table j seats at most m_j people, $i = 1, \ldots, N$, $j = 1, \ldots, M$.

Grading: Each problem 6p, total 24p.

13,067 entries
Last updated: Wed May 18 2011

Created, developed, and nurtured by Eric Weisstein at Wolfram Research

Graph Product

In general, a graph product of two graphs G and H is a new graph whose vertex set is $V(G) \times V(H)$ and where, for any two vertices (g, h) and (g', h') in the product, the adjacency of those two vertices is determined entirely by the adjacency (or equality, or non-adjacency) of g and g', and that of h and h'. There are $3 \times 3 - 1 = 8$ cases to be decided (three possibilities for each, with the case where both are equal eliminated) and thus there are $2^8 = 256$ different types of graph products that can be defined.

The most commonly used graph products, given by conditions sufficient and necessary for adjacency, are summarized in the following table (Hartnell and Rall 1998). Note that the terminology is not quite standardized, so these products may actually be referred to by different names by different sources (for example, the graph lexicographic product is also known as the graph composition; Harary 1994, p. 21). Many other graph products can be found in Jensen and Toft (1994).

graph product name	symbol	definition
graph Cartesian product	$G \square H$	$(g = g'$ and $h \operatorname{adj} h')$ or $(g \operatorname{adj} g'$ and $h = h')$
graph categorical product	$G \times H$	$(g \operatorname{adj} g'$ and $h \operatorname{adj} h')$
graph lexicographic product	$G \bullet H$	$(g \operatorname{adj} g')$ or $(g = g'$ and $h \operatorname{adj} h')$
graph strong product	$G \boxtimes H$	$(g = g'$ and $h \operatorname{adj} h')$ or $(g \operatorname{adj} g'$ and $h = h')$ or $(g \operatorname{adj} g'$ and $h \operatorname{adj} h')$

SEE ALSO: Graph Cartesian Product, Graph Categorical Product, Graph Composition, Graph Lexicographic Product, Graph Strong Product, Graph Sum