Aalto University
Department of Information and Computer Science
Petteri Kaski (tel. 23260)

T-79.5205 Combinatorics (5 cr)
Exam Mon 9 Jan 2012, 9–12 a.m.

Write down on each answer sheet:
- Your name, degree programme, and student number
- The text: "T-79.5205 Combinatorics 9.1.2012"
- The total number of answer sheets you are submitting for grading

Note: You can write down your answers in either Finnish, Swedish, or English.

1. Give a closed-form solution or a counting recurrence as a function of n and k.

   (a) Determine the number of binary strings of length n that do not contain a substring of the form "11". For example, for $n = 6$ the binary string 100101 is such a string, whereas 101100 is not. Compute the values for $1 \leq n \leq 10$.

   (b) As in part (a), but with the additional constraint that the number of "1"s in the string is exactly k. Compute the value for $n = 6$ and $0 \leq k \leq 6$.

   Hint: Start by listing the strings for small values of n.

2. Möbius inversion in partially ordered sets.

   (a) What is meant by Möbius inversion?

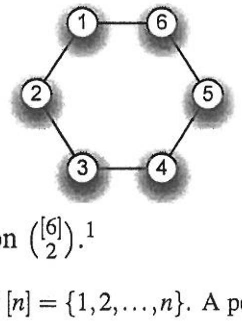
   (b) The set of all subsets of the set $[n] = \{1, 2, \ldots, n\}$ is partially ordered by the subset inclusion relation "$\subseteq$". Show that the Möbius function of this poset is given for all $A, B \subseteq [n]$ by

$$\mu(A,B) = \begin{cases} (-1)^{|B \setminus A|} & \text{if } A \subseteq B; \\ 0 & \text{otherwise.} \end{cases}$$

   Hint: Recall that every nonempty set has equally many even and odd subsets.

3. Symmetry and group actions.

   (a) Let us study the 6-cycle C_6 below. Determine the elements of the automorphism group $\mathrm{Aut}(C_6) \leq S_6$.



   (b) Draw the orbits of $\mathrm{Aut}(C_6)$ on $\binom{[6]}{2}$.[1]

[1] We write $\binom{[n]}{k}$ for the set of all k-subsets of $[n] = \{1, 2, \ldots, n\}$. A permutation σ of $[n]$ acts on $S \subseteq [n]$ by $\sigma(S) = \{\sigma(x) : x \in S\}$.

(c) For each orbit of $\mathrm{Aut}(C_6)$ on $\binom{[6]}{2}$, determine the stabilizer subgroup of the lexicographically least element of the orbit.

4. Extremal set theory.

   (a) State the Erdős–Ko–Rado Theorem.

   (b) The upper bound $\binom{n-1}{k-1}$ given by the Erdős–Ko–Rado Theorem is achieved by the families of sets containing a fixed element. Show that for $n = 2k$ there are other families achieving this bound.

   Hint: For each set, take the set itself or its complement.

   (c) Let $n \leq 2k$ and let $A_1, A_2, \ldots, A_m \subseteq [n]$ be distinct sets with $|A_i| = k$ and $A_i \cup A_j \neq [n]$ for all $1 \leq i < j \leq m$. Prove that $m \leq (1 - k/n)\binom{n}{k}$ and that equality can hold.

Grading: Each problem 12p, total 48p.