

T-61.3015 DSP Table of formulas, spring 2011

Disclaimer! Notations, e.g., ω or Ω , may vary from book to book, or from exam paper to other.

Basic math stuff

Even and odd functions:

$$\text{Even}\{x(t)\} = 0.5 \cdot [x(t) + x(-t)], \text{ e.g., } \cos(x) = \cos(-x)$$

$$\text{Odd}\{x(t)\} = 0.5 \cdot [x(t) - x(-t)], \text{ e.g., } \sin(x) = -\sin(-x)$$

Roots of second-order polynomial:

$$ax^2 + bx + c = 0, x = (-b \pm \sqrt{b^2 - 4ac})/(2a)$$

Logarithms, decibels:

$$\log((A \cdot B/C)^D) = D \cdot (\log A + \log B - \log C)$$

$$\log_a b = \log_c b / \log_c a$$

$$\text{decibels: } 10 \log_{10}(H^2/H_0^2) = 20 \log_{10}(H/H_0)$$

$$10 \log_{10}(0.5) = 20 \log_{10}(\sqrt{0.5}) \approx -3.01 \text{ dB}$$

$$20 \log_{10}(0.1) = -20 \text{ dB}, 20 \log_{10}(0.01) = -40 \text{ dB}$$

Complex numbers, radii, angles, unit circle:

$$i \equiv j = \sqrt{-1} = -1/j$$

$$z = x + jy = r e^{j\theta}$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan(y/x) + n\pi, (n=0, \text{ if } x > 0, n=1, \text{ if } x < 0)$$

$$x = r \cos(\theta), y = r \sin(\theta)$$

$$e^{j\theta} = \cos(\theta) + j \sin(\theta) \quad (\text{Euler's formula})$$

$$\cos(\theta) = (1/2) \cdot (e^{j\theta} + e^{-j\theta}), \sin(\theta) = (1/2j) \cdot (e^{j\theta} - e^{-j\theta})$$

$$z_1 \cdot z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}, z_1/z_2 = (r_1/r_2) e^{j(\theta_1 - \theta_2)}$$

$$|A \cdot B| = |A| \cdot |B|, \angle(A \cdot B) = \angle A + \angle B$$

$$z^n = r^n e^{jn\theta} = r^n (\cos \theta + j \sin \theta)^n = r^n (\cos n\theta + j \sin n\theta)$$

$$z_k = \sqrt[N]{z} = \sqrt[N]{r} e^{j(\theta+2\pi k)/N}, k = 0, 1, \dots, N-1$$

Trigonometric functions:

$$1^\circ = \pi/180 \text{ radians} \approx 0.01745 \text{ rad}, 1 \text{ rad} = 180^\circ/\pi \approx 57.30^\circ$$

$$\text{sinc}(\theta) = \sin(\pi\theta)/(\pi\theta)$$

$$\sin(\theta)/\theta \rightarrow 1, \text{ when } \theta \rightarrow 0; \text{sinc}(\theta) \rightarrow 1, \text{ when } \theta \rightarrow 0$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots + (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \dots \quad (\text{Taylor})$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots + (-1)^n \frac{\theta^{2n}}{(2n)!} + \dots \quad (\text{Taylor})$$

θ	0	$\pi/6$	$\pi/4$	$\pi/3$
$\sin(\theta)$	0	0.5	$\sqrt{2}/2$	$\sqrt{3}/2$
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	0.5
θ	$\pi/2$	$3\pi/4$	π	$-\pi/2$
$\sin(\theta)$	1	$\sqrt{2}/2$	0	-1
$\cos(\theta)$	0	$-\sqrt{2}/2$	-1	0

$$\pi \approx 3.1416, \sqrt{3}/2 \approx 0.8660, \sqrt{2}/2 \approx 0.7071$$

Geometric series:

$$\sum_{n=0}^{+\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

$$\sum_{n=0}^N a^n = \frac{1-a^{N+1}}{1-a}, \quad |a| < 1$$

Continuous-time unit step and unit impulse fun.:

$$\mu(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

$$\delta_\Delta(t) = \frac{d}{dt} \mu_\Delta(t), \quad \delta(t) = \lim_{\Delta \rightarrow 0} \delta_\Delta(t) \quad (\text{Dirac's delta})$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t-t_0)x(t) dt = x(t_0)$$

In DSP notation $2\pi\delta(t)$ is computed $2\pi \int \delta(t) \cdot 1 dt = 2\pi$, when $t = 0$, and = 0 elsewhere.

Discrete-time unit impulse and unit step functions:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \quad \mu[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$\text{E.g.: } x[n] = 2\delta[n+1] + \delta[n] - \delta[n-1] = \{2, 1, -1\}, x[-1] = 2, x[0] = 1, x[1] = -1.$$

Periodic signals

$$\exists T \in \mathbb{R}_+: x(t) = x(t+T), \forall t \in \mathbb{R}$$

$$\exists N \in \mathbb{Z}_+: x[n] = x[n+N], \forall n \in \mathbb{Z}$$

Fundamental period T_0 , N_0 is the smallest $T > 0$, $N > 0$.

$$\text{E.g.: } \cos(\omega n) = \cos(\omega n + 2\pi k), e^{j(\omega n)} = e^{j(\omega n + 2\pi k)}$$

Convolution

Convolution is commutative, associative and distributive.

$$y(t) = h(t) \circledast x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau) d\tau$$

$$y[n] = h[n] \circledast x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$y_C[n] = h[n] \otimes x[n] = \sum_{k=0}^{N-1} h[k]x[n-k]$$

Correlation:

$$r_{xy}[l] = \sum_{n=-\infty}^{+\infty} x[n]y[n-l] = x[l] \circledast y[-l]$$

$$r_{xx}[l] = \sum_{n=-\infty}^{+\infty} x[n]x[n-l]$$

Mean and variance of random signal:

$$m_X = E[X] = \int x p_X(x) dx$$

$$\sigma_X^2 = \int (x - m_X)^2 p_X(x) dx = E[X^2] - m_X^2$$

Frequencies, angular frequencies, periods:

Here f_s (also f_T later) is the sampling frequency.

$$\text{Frequency } f, [f] = \text{Hz} = 1/\text{s}.$$

$$\text{Angular frequency } \Omega = 2\pi f = 2\pi/T, [\Omega] = \text{rad/s (analog)}$$

$$\text{Normalized angular frequency } \omega = 2\pi\Omega/\Omega_s = 2\pi f/f_s, [\omega] = \text{rad/sample (digital).}$$

$$\text{Normalized frequency in Matlab } f_{MATLAB} = 2f/f_s, [f_{MATLAB}] = 1/\text{sample.}$$

Sampling of $x_a(t)$ by sampling frequency f_T

$$x_p[n] = x_a(nT) = x_a(n/f_T)$$

$$X_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(j(\Omega - k\Omega_T))$$

Integral transforms. Properties

Here all integral transforms share some basic properties.

Examples given with CTFT, $x[n] \leftrightarrow X(e^{j\omega})$, $x_1[n] \leftrightarrow X_1(e^{j\omega})$, and $x_2[n] \leftrightarrow X_2(e^{j\omega})$ are time-domain signals with corresponding transform-domain spectra. a and b are constants.

Linearity. All transforms are linear.

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

Time-shifting. There is a kernel term in transform, e.g., $x[n-k] \leftrightarrow e^{-jk\omega} X(e^{j\omega})$

Frequency-shifting. There is a kernel term in signal e.g., $e^{j\omega_k n} x[n] \leftrightarrow X(e^{j(\omega-\omega_k)})$

Conjugate symmetry $x^*[n] \leftrightarrow X^*(e^{-j\omega})$. If $x[n] \in \mathbb{R}$, then $X(e^{j\omega}) = X^*(e^{-j\omega})$, $|X(e^{j\omega})| = |X(e^{-j\omega})|$, $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$. If $x[n] \in \mathbb{R}$ and even, then $X(e^{j\omega}) \in \mathbb{R}$ and even. If $x[n] \in \mathbb{C}$ and odd, then $X(e^{j\omega})$ purely $\in \mathbb{C}$ and odd.

Time reversal. Transform variable is reversed, e.g., $x[-n] \leftrightarrow X(e^{-j\omega})$

Differentiation. In time and frequency domain, e.g., $x[n] - x[n-1] \leftrightarrow (1 - e^{-j\omega}) X(e^{j\omega})$, $nx[n] \leftrightarrow j \frac{d}{d\omega} X(e^{j\omega})$

Duality. Convolution property: convolution in time domain corresponds multiplication in transform domain $x_1[n] \circledast x_2[n] \leftrightarrow X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$ and multiplication property, vice versa, $x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$

Parseval's relation. Energy in signal and spectral components: $\sum |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$

Fourier series of continuous-time periodic signals:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \quad (\text{synthesis})$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\Omega_0 t} dt \quad (\text{analysis})$$

$$x(t-t_0) \leftrightarrow a_k e^{jk\Omega_0 t_0}$$

$$e^{jM\Omega_0 t} x(t) \leftrightarrow a_{k-M}$$

$$\int_T x_a(\tau) x_b(t-\tau) d\tau \leftrightarrow T a_k b_k$$

$$x_a(t) x_b(t) \leftrightarrow \sum_l a_l b_{k-l}$$

$$\frac{d}{dt} x(t) \leftrightarrow jk\Omega_0 a_k$$

Continuous-time Fourier-transform (CTFT):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \quad (\text{synthesis})$$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \quad (\text{analysis})$$

$$x(t-t_k) \leftrightarrow e^{j\Omega t_k} X(j\Omega)$$

$$\begin{aligned}
e^{j\Omega_k t} x(t) &\leftrightarrow X(j(\Omega - \Omega_k)) \\
x_a(t) \oplus x_b(t) &\leftrightarrow X_a(j\Omega)X_b(j\Omega) \\
x_a(t)x_b(t) &\leftrightarrow \frac{1}{2\pi} X_a(j\Omega) \otimes X_b(j\Omega) \\
\frac{d}{dt} x(t) &\leftrightarrow j\Omega X(j\Omega) \\
tx(t) &\leftrightarrow j \frac{d}{d\Omega} X(j\Omega) \\
e^{j\Omega_0 t} &\leftrightarrow 2\pi\delta(\Omega - \Omega_0) \\
\cos(\Omega_0 t) &\leftrightarrow \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)] \\
\sin(\Omega_0 t) &\leftrightarrow j\pi[\delta(\Omega + \Omega_0) - \delta(\Omega - \Omega_0)] \\
x(t) = 1 &\leftrightarrow 2\pi\delta(\Omega) \\
x(t) = \begin{cases} 1, |t| < T_1 \\ 0, |t| > T_1 \end{cases} &\leftrightarrow \frac{2\sin(\Omega T_1)}{\Omega} \\
\frac{\sin(Wt)}{\pi t} &\leftrightarrow X(j\Omega) = \begin{cases} 1, |\Omega| < W \\ 0, |\Omega| > W \end{cases} \\
\delta(t) &\leftrightarrow 1 \\
\delta(t - t_k) &\leftrightarrow e^{j\Omega t_k} \\
e^{-at}\mu(t) &\leftrightarrow \frac{1}{a+j\Omega}, \text{ where } \operatorname{Real}\{a\} > 0
\end{aligned}$$

Discrete-time Fourier-transform (DTFT):

$$\begin{aligned}
x[n] &= \frac{1}{2\pi} \int_{-2\pi}^{2\pi} X(e^{j\omega}) e^{jn\omega} d\omega \quad (\text{synthesis}) \\
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-jn\omega}, \quad \text{periodic with } 2\pi \quad (\text{analysis}) \\
a x[n-k] &\leftrightarrow a e^{-jk\omega} X(e^{j\omega}) \\
e^{j\omega_k n} x[n] &\leftrightarrow X(e^{j(\omega-\omega_k)}) \\
x_1[n] \oplus x_2[n] &\leftrightarrow X_1(e^{j\omega}) \cdot X_2(e^{j\omega}) \\
x_1[n] \cdot x_2[n] &\leftrightarrow \frac{1}{2\pi} \int_{-2\pi}^{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta \\
nx[n] &\leftrightarrow j \frac{d}{d\omega} X(e^{j\omega}) \\
e^{j\omega_0 n} &\leftrightarrow 2\pi \sum_l \delta(\omega - \omega_0 - 2\pi l) \\
\cos(\omega_0 n) &\leftrightarrow \pi \sum_l [\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)] \\
\sin(\omega_0 n) &\leftrightarrow j\pi \sum_l [\delta(\omega + \omega_0 - 2\pi l) - \delta(\omega - \omega_0 - 2\pi l)] \\
x[n] = 1 &\leftrightarrow 2\pi \sum_l \delta(\omega - 2\pi l) \\
x[n] = \begin{cases} 1, |n| \leq N_1 \\ 0, |n| > N_1 \end{cases} &\leftrightarrow \frac{\sin(\omega(N_1+0.5))}{\sin(\omega/2)} \\
\frac{\sin(Wn)}{\pi n} = \frac{W}{\pi} \operatorname{sinc}(\frac{Wn}{\pi}) &\leftrightarrow X(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq W \\ 0, & W < |\omega| \leq \pi \end{cases} \\
\delta[n] &\leftrightarrow 1 \\
a \delta[n-k] &\leftrightarrow a e^{-jk\omega} \\
a^n \mu[n] &\leftrightarrow \frac{1}{1-ae^{-j\omega}}, \quad |a| < 1
\end{aligned}$$

N-point Discrete Fourier-transform (DFT):

$$\begin{aligned}
\text{Connection to DTFT: } X[k] &= X(e^{j\omega})|_{\omega=2\pi k/N} \\
W_N &= e^{-j2\pi/N} \\
x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \leq n \leq N-1 \quad (\text{synthesis}) \\
X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \leq k \leq N-1 \quad (\text{analysis}) \\
x[< n - n_0 >_N] &\leftrightarrow W_N^{-kn_0} X[k] \\
W_N^{-kn_0} x[n] &\leftrightarrow X[< k - k_0 >_N] \\
y_C[n] &= h[n] \otimes x[n] \leftrightarrow H[k] \cdot X[k] = Y[k]
\end{aligned}$$

Laplace transform:

$$\begin{aligned}
\text{Convergence with a certain ROC (region of convergence).} \\
\text{Connection to continuous-time Fourier-transform: } s = j\Omega \\
x(t) &= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds \quad (\text{synthesis}) \\
X(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt \quad (\text{analysis})
\end{aligned}$$

z-transform:

$$\begin{aligned}
\text{Convergence with a certain ROC (region of convergence).} \\
\text{Connection to discrete-time Fourier-transform: } z = e^{j\omega} \\
x[n] &= \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz, \quad C \text{ in ROC of } X(z) \quad (\text{synthesis}) \\
X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad (\text{analysis}) \\
a x[n-k] &\leftrightarrow a z^{-k} X(z) \\
x_1[n] \oplus x_2[n] &\leftrightarrow X_1(z) \cdot X_2(z) \\
\delta[n] &\leftrightarrow 1, \quad \text{ROC all } z \\
a \delta[n-k] &\leftrightarrow a z^{-k}, \quad \text{all } z, \text{ except } 0 \ (k > 0) \text{ or } \infty \ (k < 0) \\
\mu[n] &\leftrightarrow \frac{1}{1-z^{-1}}, \quad |z| > 1 \\
-\mu[-n-1] &\leftrightarrow \frac{1}{1-z^{-1}}, \quad |z| < 1 \\
a^n \mu[n] &\leftrightarrow \frac{1}{1-az^{-1}}, \quad |z| > |a| \\
na^n \mu[n] &\leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|
\end{aligned}$$

$$\begin{aligned}
(n+1)a^n \mu[n] &\leftrightarrow \frac{1}{(1-az^{-1})^2}, \quad |z| > |a| \\
r^n \cos(\omega_0 n) \mu[n] &\leftrightarrow \frac{1-r \cos(\omega_0) z^{-1}}{1-2r \cos(\omega_0) z^{-1} + r^2 z^{-2}}, \quad |z| > |r| \\
r^n \sin(\omega_0 n) \mu[n] &\leftrightarrow \frac{r \sin(\omega_0) z^{-1}}{1-2r \cos(\omega_0) z^{-1} + r^2 z^{-2}}, \quad |z| > |r|
\end{aligned}$$

LTI filter analysis

Stability $\sum_n |h[n]| < \infty$; unit circle belongs to ROC

Causality $h[n] = 0, n < 0$; ∞ belongs to ROC

Unit step response $s[n] = \sum_{k=-\infty}^n h[k]$

Causal transfer function of order $\max\{M, N\}$:

$$H(z) = B(z)/A(z) = K \cdot \frac{\sum_{m=0}^M b_m z^{-m}}{\sum_{n=0}^N a_n z^{-n}} = G \cdot \frac{\prod_{m=1}^M (1-d_m z^{-1})}{\prod_{n=1}^N (1-p_n z^{-1})}$$

Zeros d_m : $B(z) = 0$; *Poles* p_n : $A(z) = 0$

Frequency, magnitude/amplitude, phase response, $z \leftarrow e^{j\omega}$

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$$

$$H[k] = H(e^{j\omega})|_{\omega=2\pi k/N}$$

$$\text{Group delay } \tau(\omega) = -\frac{d}{d\omega} \angle H(e^{j\omega})$$

Four types of linear-phase FIR filters, $h[n] = h[N-1-n]$ (even/odd symmetric), or $h[n] = -h[N-1-n]$ (e/o antis.). Zeros symmetric w.r.t. unit circle: $r e^{\pm j\theta}$ and $(1/r) e^{\mp j\theta}$.

Important transform pairs and properties:

$$a \delta[n-k] \leftrightarrow a e^{-jk\omega} \leftrightarrow a z^{-k}$$

$$a^n \mu[n] \leftrightarrow 1/[1 - a e^{-j\omega}] \leftrightarrow 1/[1 - a z^{-1}]$$

$$h[n] = \sum_i (k_i \cdot a_i^n \mu[n]) \leftrightarrow H(e^{j\omega}) = \dots$$

$$\dots \sum_i (k_i/[1 - a_i e^{-j\omega}]) \leftrightarrow H(z) = \sum_i (k_i/[1 - a_i z^{-1}])$$

$$a x[n-k] \leftrightarrow a e^{-jk\omega} X(e^{j\omega}) \leftrightarrow a z^{-k} X(z)$$

$$y[n] = h[n] \otimes x[n] \leftrightarrow Y(z) = H(z) \cdot X(z)$$

rectangle \leftrightarrow sinc, sinc \leftrightarrow rectangle

LTI filter design (synthesis)

Bilinear transform $H(z) = H(s)|_s$ and *prewarping*

$$s = k \cdot (1 - z^{-1})/(1 + z^{-1}), \quad k = 1 \text{ or } k = 2/T = 2f_T$$

$$\Omega_{\text{prewarp},c} = k \cdot \tan(\omega_c/2), \quad k = 1 \text{ or } k = 2/T = 2f_T$$

Spectral transformations, $\hat{\omega}_c$ desired cut-off

$$\text{LP-LP } z^{-1} = (z^{-1} - \alpha)/(1 - \alpha z^{-1}), \text{ where}$$

$$\alpha = \sin(0.5(\omega_c - \hat{\omega}_c))/\sin(0.5(\omega_c + \hat{\omega}_c))$$

$$\text{LP-HP } z^{-1} = -(\hat{z}^{-1} + \alpha)/(1 + \alpha \hat{z}^{-1}), \text{ where}$$

$$\alpha = -\cos(0.5(\omega_c + \hat{\omega}_c))/\cos(0.5(\omega_c - \hat{\omega}_c))$$

Windowed Fourier series method

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| \geq \omega_c \end{cases} \leftrightarrow h[n] = \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi} \operatorname{sinc}(\frac{\omega_c n}{\pi})$$

$$h_{\text{FIR}}[n] = h_{\text{ideal}}[n] \cdot w[n]$$

$$H_{\text{FIR}}(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h_{\text{ideal}}(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta$$

Fixed window functions, order $N = 2M$, $-M \leq n \leq M$:

Rectangular $w[n] = 1$

Hamming $w[n] = 0.54 + 0.46 \cos((2\pi n)/(2M))$

Hann $w[n] = 0.5 \cdot (1 + \cos((2\pi n)/(2M)))$

Blackman $w[n] = 0.42 + 0.5 \cos(\frac{2\pi n}{2M}) + 0.08 \cos(\frac{4\pi n}{2M})$

Bartlett $w[n] = 1 - (|n|/M)$

Implementation

Radix-2 DIT N-FFT butterfly equations, $L\{x[n]\} = N$

$$\begin{cases} \Psi_{r+1}[\alpha] = \Psi_r[\alpha] + W_{N_r}^l \Psi_r[\beta] \\ \Psi_{r+1}[\beta] = \Psi_r[\alpha] - W_{N_r}^l \Psi_r[\beta] \end{cases}$$

Inputs $x[n]$ in bit-reversed order in Ψ_1 , outputs in Ψ_R , $R = \log_2 N + 1$; levels Ψ_1, \dots, Ψ_R , $r \in [1, R-1]$; multipliers $W_{N_r} = e^{-j2\pi/N_r}$, $N_r = 2^r$, $l \in [0, 2^{r-1} - 1]$.

Multirate systems

Upsampling (interpolation) with factor L, $\uparrow L$

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ x_u[n] = 0, & \text{otherwise} \end{cases}$$

$$X_u(z) = X(z^L), \quad X_u(e^{j\omega}) = X(e^{j\omega L})$$

Downsampling (decimation) with factor M, $\downarrow M$

$$x_d[n] = x[nM]$$

$$X_d(z) = (1/M) \sum_{k=0}^{M-1} X(z^{1/M} W_M^{-k})$$

$$X_d(e^{j\omega}) = (1/M) \sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$$