Mathematics
Aalto University

Kibble/Metsalo

Mat-1.1620 Mathematics 2

Final exam, 1 Sep 2011

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination. Degree Programmes are ARK, AUT, BIO, EST, ENE, GMA, INF, KEM, KTA, KON, MAR, MTE, PUU, RRT, TFM, TIK, TLT, TUO, YYT.

Calculators are not allowed. The examination time is 4 hours.

1. a) Solve the following initial value problem for $x > 0$:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x^4}{12y^2}, \quad y(1) = 1.$$

   b) Find the general solution for the differential equation $y'' + y' - 2y = e^{-x}$.

2. a) Parametrize the curve of intersection of the surfaces $x^2 + y^2 = 9$ and $z = x + y$.
   b) Find and classify the critical points of the function $f(x, y) = x^3 + y^3 - 3xy$.

3. A circular disk with radius R and hence area $A = \pi R^2$ has area-density $\delta(r) = \delta_0 \cdot \cos(\frac{\pi r}{2R})$ at the distance $r \leq R$ from its center, so the area-density is greatest at the center and diminishes towards the boundary. Calculate the average area-density (mass/area) of the disk.

4. For vectors $\vec{a}, \vec{b}$ and $\vec{c}$ in $\mathbf{R}^3$ we have that

$$\vec{a} \bullet (\vec{b} \times \vec{c}) = \vec{b} \bullet (\vec{c} \times \vec{a}) = \vec{c} \bullet (\vec{a} \times \vec{b}).$$

   The corresponding formula does not hold, if ∇ is involved. Show that if $\vec{F}$ and $\vec{G}$ are two vectorfields of class $C^1(\mathbf{R}^3)$, then

$$\nabla \bullet (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \bullet \vec{G} - \vec{F} \bullet (\nabla \times \vec{G}), \text{ i.e. } div(\vec{F} \times \vec{G}) = (curl(\vec{F})) \bullet \vec{G} - \vec{F} \bullet (curl(\vec{G})).$$

5. Green's theorem states that

$$\oint_{\partial R} \vec{F} \bullet d\vec{r} = \oint_{\partial R} (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy,$$

   if we integrate around the boundary ∂R counterclockwise. Verify Green's theorem for the case when the vectorfield $\vec{F}(x, y) = 2x\vec{\imath} + xy\vec{\jmath}$ and the plane region R is bounded by the line $y = 1$ and the parabola $y = 2x^2 - 1$, as in the figure below.