10.2.2009

Tfy-44.130 KVANTTIMEKANIIKKA II Whole course exam [redacted]

1. (a) Schrödinger and Heisenberg pictures; transformation between these and the equations of motion.

   (b) Suppose that we have a physical system, and we measure a quantity that corresponds to operator $\hat{A} = \hat{A}_S$ in the Schrödinger picture. Suppose that $\hat{A}_S$ is time-independent, and has a nondegenerate eigenvalue λ. Let $|\psi_S(0)\rangle$ be the Schrödinger picture state vector of the system at time $t = 0$. Show explicitly that the probability to obtain the measurement result λ at time $t > 0$ is exactly the same in the Schrödinger and Heisenberg pictures.

2. Calculate the phase shift δ_0 for s-wave scattering for an attractive square-well potential of the form

$$V(r) = \begin{cases} -V_0 & \text{for } r < a, \\ 0 & \text{for } r > a \end{cases}$$

in the low-energy limit $ka \ll 1$. Do you find scattering resonances?

(Hints: The s-wave states are exactly of the form $\sin(kr + \delta_0)/kr$ in the region $r > a$ for this kind of potential that vanishes outside radius a. You may also find the relation $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2}(\mathbf{r} \times \nabla)^2$ useful.)

3. Consider the two-particle boson state

$$|2\rangle = \int d\mathbf{x}_1 \int d\mathbf{x}_2\, \phi(\mathbf{x}_1, \mathbf{x}_2)\psi^\dagger(\mathbf{x}_1)\psi^\dagger(\mathbf{x}_2)|0\rangle,$$

where $\psi(\mathbf{x})$ is the bosonic field operator.

   (a) Show that the norm squared of this state is

$$\langle 2|2\rangle = \int d\mathbf{x}_1 \int d\mathbf{x}_2\, \phi^*(\mathbf{x}_1, \mathbf{x}_2)[\phi(\mathbf{x}_1, \mathbf{x}_2) + \phi(\mathbf{x}_2, \mathbf{x}_1)].$$

   We choose the wave function $\phi(\mathbf{x}_1, \mathbf{x}_2)$ such that this integral equals unity, i.e. then $\langle 2|2\rangle = 1$.

   (b) Assuming that the wavefunction is of the uncorrelated form $\phi(\mathbf{x}_1, \mathbf{x}_2) = \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)$, calculate the expectation value $\langle 2|\hat{n}(\mathbf{x})|2\rangle$ of particle density operator in this state. Present the result in terms of ϕ_1 and ϕ_2. Comment/interpret the result.

(continues...)

[redacted]

4. (a) Present the Dirac equation for a particle moving in an external central potential $\phi(\mathbf{x}) = \phi(r)$ in the form of a (time-dependent) Schrödinger equation (Hint: If you do not recall the precise form of the constant coefficients, you can infer them from dimensional analysis...). What is the Hamiltonian operator $\hat{H}$ corresponding to the Dirac equation? What can the Dirac equation be used for?

(b) Calculate the energy spectrum (energy eigenvalues) for a free particle described by the Dirac equation!

[Hints: The free particle Dirac Hamiltonian commutes with momentum operator $\hat{\mathbf{p}}$, hence you may require that the energy eigenstates are also eigenstates of the momentum operator. Use thus the Ansatz $\psi(\mathbf{r}, t) = e^{-ip_\mu x^\mu/\hbar} v(p)$, where $p^\mu = (E/c, \mathbf{p})$, $x^\mu = (ct, \mathbf{x})$ are the energy-momentum and space-time four-vectors, and

$$v(p) = \begin{pmatrix} \xi(p) \\ \eta(p) \end{pmatrix},$$

where $\xi(p)$ and $\eta(p)$ are both 2×1 column vectors. The relation $\sigma^j \sigma^k = \delta_{jk} + i\epsilon^{jkl}\sigma^l$ could also be useful.]