Aalto University, School of Science and Technology — **Exam 18.05.2011**
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Mat-1.3601 Introduction to Stochastics, Spring 2011

1. Let X_1 and X_2 be two independent exponentially distributed random variables with parameters λ_1 and λ_2 respectively. Show that

$$\mathbb{P}(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

2. For two random variables X and Y define the distance

$$d(X,Y) = \mathbb{E}\Big(\frac{|X-Y|}{1+|X-Y|}\Big).$$

Show that $X_n \to X$ in probability if and only if $d(X_n, X) \to 0$ as n tends to infinity.

3. Let X be a standard Gaussian random variable and $a > 0$. Define

$$Z = \begin{cases} X & \text{if } |X| \le a, \\ -X & \text{if } |X| > a. \end{cases}$$

What is distribution of random variable Z ? Justify your answer.

4. Let $X_1, X_2, \cdots$ be independent, nonnegative and identically distributed random variables with $\mathbb{E}(X_1) = 1$ and $\mathrm{Var}(X_1) = \sigma^2 \in (0, \infty)$. Show that

$$\frac{2}{\sigma}(\sqrt{S_n} - \sqrt{n}) \longrightarrow Z$$

weakly as n tends to infinity, where Z is a random variable with distribution $N(0,1)$. Here S_n stands for partial sum $S_n = \sum_{i=1}^n X_i$.
[Hint: $a - b = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}), \quad a, b > 0$].

5. Let $X_1, X_2, \cdots$ be independent, nonnegative and identically distributed random variables with $\mathbb{E}(X_1) = 1$. Show that

$$M_n = \prod_{i \le n} X_i$$

is an $\mathcal{F}_n$ martingale, where $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$.