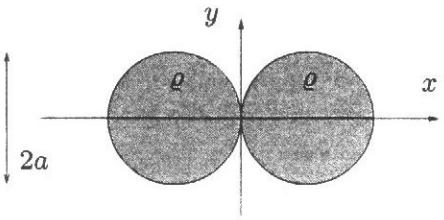
— Laske tehtävät 1 + 2 eri papereille kuin tehtävät 3 + 4 —

TASKULASKIN SALLITTU, EI APUKIRJALLISUUTTA.

1. Tyhjässä avaruudessa on kaksi toisiaan hipaisevaa varauspalloa (säde a) oheisen kuvan mukaisesti. Varausjakautuma on niissä tasainen $\varrho = Q/(4\pi a^3/3)$.



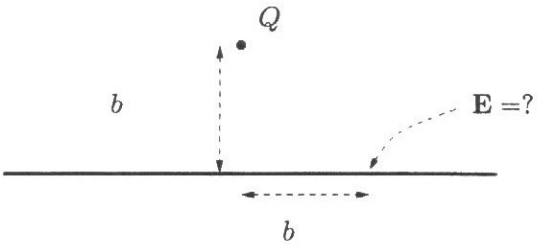
   (a) Laske sähkökenttä $\mathbf{E}$ (suunta ja suuruus) pisteessä

      i. $x = a, y = z = 0$ (siis toisen pallon keskipisteessä)

      ii. $x = y = z = a$

   (b) Määritä kaikki paikat, joissa kenttä häviää, $\mathbf{E} = 0$.

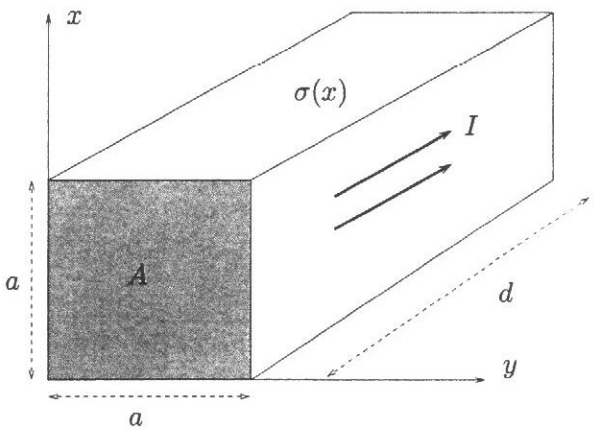
2. Pistevaraus Q on korkeudella b maadoitetusta johdetasosta. Laske sähkökenttävektori kuvan osoittamassa kohdassa johteen pinnalla. Kuinka paljon sähkökenttä ko. paikassa muuttuu (suuruudeltaan ja suunnaltaan), jos johdetaso poistetaan?



3. Suorakulmainen vastus, jonka pituus on $d = 10\,\mathrm{cm}$, ja poikkipinta-ala $A = a \times a = 2\,\mathrm{cm} \times 2\,\mathrm{cm}$ on täytetty epähomogeenisella materiaalilla, jonka johtavuus muuttuu lineaarisesti vastuksen poikittaisen etäisyyden x funktiona arvosta $\sigma_1 = 1\,\mathrm{S/m}$ arvoon $\sigma_2 = 5\,\mathrm{S/m}$ kaavan

$$\sigma(x) = \sigma_1 + \frac{\sigma_2 - \sigma_1}{a} x$$

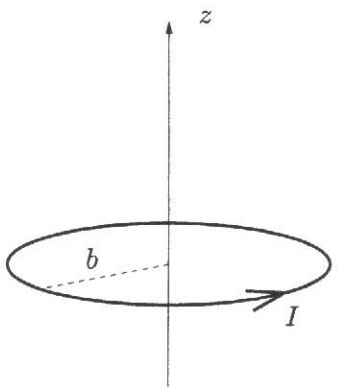
mukaisesti. Vastuksen läpi kulkevan virrantiheyden oletetaan olevan joka kohdassa z:n suuntainen: $\mathbf{J}(\mathbf{r}) = \mathbf{u}_z J(\mathbf{r})$. Laske vastuksen resistanssi R.



4. Biot–Savartin lain mukaan differentiaalinen tasavirtaalkio $I d\mathbf{l}$ synnyttää etäisyydelle D magneettikentän

$$d\mathbf{H} = \frac{I\, d\mathbf{l} \times \mathbf{u}_D}{4\pi D^2}$$

Laske sen perusteella b-säteisen ympyränmuotoisen virtasilmukan, jossa kulkee virta I, magneettikenttä $\mathbf{H}$ kahdessa pisteessä: silmukan keskellä origossa ($x = y = z = 0$) ja silmukan akselilla, sen yläpuolella kohdassa $x = y = 0, z = b$.



Nablaoperaatiot

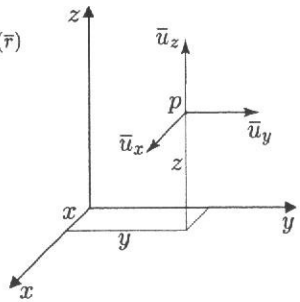
Karteesinen koordinaatisto

$$\nabla f(\overline{r}) = \overline{u}_x \frac{\partial}{\partial x} f(\overline{r}) + \overline{u}_y \frac{\partial}{\partial y} f(\overline{r}) + \overline{u}_z \frac{\partial}{\partial z} f(\overline{r})$$

$$\nabla \times \overline{f} = \begin{vmatrix} \overline{u}_x & \overline{u}_y & \overline{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$\nabla \cdot \overline{f}(\overline{r}) = \frac{\partial}{\partial x} f_x(\overline{r}) + \frac{\partial}{\partial y} f_y(\overline{r}) + \frac{\partial}{\partial z} f_z(\overline{r})$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$



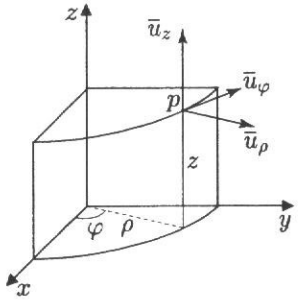
Sylinterikoordinaatisto

$$\nabla f(\overline{r}) = \overline{u}_\rho \frac{\partial}{\partial \rho} f + \overline{u}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} f + \overline{u}_z \frac{\partial}{\partial z} f$$

$$\nabla \times \overline{f} = \frac{1}{\rho} \begin{vmatrix} \overline{u}_\rho & \rho \overline{u}_\varphi & \overline{u}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ f_\rho & \rho f_\varphi & f_z \end{vmatrix}$$

$$\nabla \cdot \overline{f} = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho f_\rho) + \frac{1}{\rho} \frac{\partial}{\partial \varphi} f_\varphi + \frac{\partial}{\partial z} f_z$$

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$



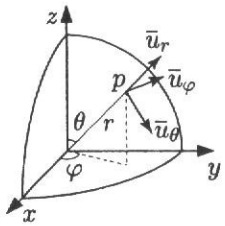
Pallokoordinaatisto

$$\nabla f(\overline{r}) = \overline{u}_r \frac{\partial}{\partial r} f + \overline{u}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} f + \overline{u}_\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} f$$

$$\nabla \times \overline{f} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \overline{u}_r & r\overline{u}_\theta & r\sin\theta \overline{u}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ f_r & r f_\theta & r \sin\theta f_\varphi \end{vmatrix}$$

$$\nabla \cdot \overline{f} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta f_\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} f_\varphi$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \varphi^2}$$



Koordinaattimuunnokset vektorille $\overline{f}$

Karteesinen ↔ sylinterikoordinaatisto

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z,$$

$$\rho = \sqrt{x^2 + y^2}, \quad \varphi = \arctan(y/x), \quad z = z.$$

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_\rho \\ f_\varphi \\ f_z \end{pmatrix},$$

$$\begin{pmatrix} f_\rho \\ f_\varphi \\ f_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}.$$

Karteesinen ↔ pallokoordinaatisto

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta,$$

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right), \quad \varphi = \tan^{-1}\frac{y}{x}$$

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} f_r \\ f_\theta \\ f_\varphi \end{pmatrix},$$

$$\begin{pmatrix} f_r \\ f_\theta \\ f_\varphi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}.$$

Sylinteri ↔ pallokoordinaatisto

$$\rho = r\sin\theta, \quad \varphi = \varphi, \quad z = r\cos\theta,$$

$$r = \sqrt{\rho^2 + z^2}, \quad \theta = \arctan(\rho/z), \quad \varphi = \varphi.$$

$$\begin{pmatrix} f_\rho \\ f_\varphi \\ f_z \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} f_r \\ f_\theta \\ f_\varphi \end{pmatrix},$$

$$\begin{pmatrix} f_r \\ f_\theta \\ f_\varphi \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_\rho \\ f_\varphi \\ f_z \end{pmatrix}.$$

Vektori-integraalilaskennan kaavoja

Karteesinen koordinaatisto

$$\overline{d\ell} = \overline{u}_x\, dx + \overline{u}_y\, dy + \overline{u}_z\, dz$$

$$\overline{dS_x} = \overline{u}_x\, dy\, dz$$

$$\overline{dS_y} = \overline{u}_y\, dx\, dz$$

$$\overline{dS_z} = \overline{u}_z\, dx\, dy$$

$$dV = dx\, dy\, dz$$

Sylinterikoordinaatisto

$$\overline{d\ell} = \overline{u}_\rho\, d\rho + \overline{u}_\varphi \rho\, d\varphi + \overline{u}_z\, dz$$

$$\overline{dS_\rho} = \overline{u}_\rho \rho\, d\varphi\, dz$$

$$\overline{dS_\varphi} = \overline{u}_\varphi\, d\rho\, dz$$

$$\overline{dS_z} = \overline{u}_z \rho\, d\rho\, d\varphi$$

$$dV = \rho\, d\rho\, d\varphi\, dz$$

Pallokoordinaatisto

$$\overline{d\ell} = \overline{u}_r\, dr + \overline{u}_\theta r\, d\theta + \overline{u}_\varphi r \sin\theta\, d\varphi$$

$$\overline{dS_r} = \overline{u}_r r^2 \sin\theta\, d\theta\, d\varphi$$

$$\overline{dS_\theta} = \overline{u}_\theta r \sin\theta\, dr\, d\varphi$$

$$\overline{dS_\varphi} = \overline{u}_\varphi r\, dr\, d\theta$$

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

Gaussin lause $\displaystyle\int_V \nabla \cdot \overline{f}\, dV = \oint_S \overline{f} \cdot \overline{dS}$

Stokesin lause $\displaystyle\int_S \nabla \times \overline{f} \cdot \overline{dS} = \oint_C \overline{f} \cdot \overline{d\ell}$

Vakioita

$$\epsilon_0 = 8.854 \cdot 10^{-12}\ \frac{\mathrm{As}}{\mathrm{Vm}}$$

$$\mu_0 = 4\pi \cdot 10^{-7}\ \frac{\mathrm{Vs}}{\mathrm{Am}}$$

$$k_B = 1.38 \cdot 10^{-23}\ \frac{\mathrm{J}}{\mathrm{K}}$$

$$e = 1.60 \cdot 10^{-19}\ \mathrm{C}$$