Department of Mathematics and Systems Analysis
Mat-1.3601 Introduction to Stochastics, Spring 2012

Final exam **16.5.2012** **Tikanmäki**

Solve all problems. You can answer either in English or Finnish.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

1. Let $(A_j)_{j=1}^n$ be a sequence of null sets. Show that $\cup_{j=1}^n A_j$ is also a null set.

2. Let X and Y be independent and identically distributed. Show that $Z = X - Y$ has symmetric distribution (i.e. $\mathbb{P}^Z = \mathbb{P}^{-Z}$).

3. Let $(X_n)_{n\geq 1}$ be a sequence of random variables. Show that

$$X_n \to_{n\to\infty} 0,$$

in probability if and only if

$$\mathbb{E}(\min(|X_n|, 1)) \to_{n\to\infty} 0.$$

4. Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables with mean μ and variance σ^2. Show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (X_j - \mu)^2 = \sigma^2 \quad a.s.$$

5. Let $X \in L^1$. Show that $\mathbb{E}(X|X) = X$. In what sense is this conditional expectation unique? (Recall that $\mathbb{E}(Y|X) = \mathbb{E}(Y|\sigma(X))$.)