ALGEBRA 1
19.5.2012 10:00-13:00 (3h)

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1. Define/explain the following concepts:

a) Group and normal subgroup.

b) Ideal.

c) Ring homomorphism and isomorphism.

2. Let $G = \mathbf{Z}_8$, and $H = < 4 >$.

a) List the distinct left cosets of H in G.

b) What is the order of $6 + H$ in G/H?

c) To what group is G/H isomorphic to?

3.

a) State and prove *Lagrange's theorem*.

b) Let $f : G \to G'$ be a group homomorphism. Show that

$$N' \trianglelefteq G' \Rightarrow f^{-1}(N') \trianglelefteq G.$$

You can assume $f^{-1}(N') \leq G$.

4.

a) Let R be a ring. The center of R is defined as $\mathscr{Z}(R) = \{x \in R \,|\, xy = yx \ \ \forall y \in R\}$. Prove that $\mathscr{Z}(R)$ is a subring of R.

b) State the *Homomorphism theorem* of rings. What does it tell about the interrelation of the rings $\mathbf{Z}$, $\mathbf{Z}_m$ and $\mathbf{Z}/m\mathbf{Z}$?

5.

a) Prove that any finite integral domain is a field.

b) Define the set
$$K = \mathbf{Q}[x]/\langle x^2 - 2 \rangle.$$
Show that K is a field, and that it is isomorphic to $K' = \mathbf{Q}(\sqrt{2})$.