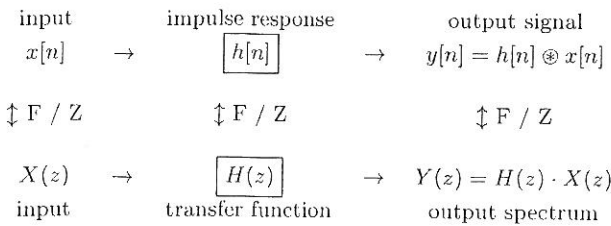


T-61.3015 DSP Table of formulas, spring 2012

**Disclaimer!** Notations, e.g.,  $\omega$  or  $\Omega$ , may vary from book to book, or from exam paper to other.

**DSP context**, change between time (1st row) and frequency (2nd row) domain using Fourier / Z-transform.



**Basic math stuff**

*Even and odd functions*

*Even*  $\{x(t)\} = 0.5 \cdot [x(t) + x(-t)]$ , e.g.,  $\cos(x) = \cos(-x)$

*Odd*  $\{x(t)\} = 0.5 \cdot [x(t) - x(-t)]$ , e.g.,  $\sin(x) = -\sin(-x)$

*Roots of second-order polynomial*

$$ax^2 + bx + c = 0, x = (-b \pm \sqrt{b^2 - 4ac}) / (2a)$$

*Logarithms, decibels*

$$\log((A \cdot B/C)^D) = D \cdot (\log A + \log B - \log C)$$

$$\log_a b = \log_r b / \log_r a$$

$$\text{decibels } 10 \log_{10}(H^2/H_0^2) = 20 \log_{10}(H/H_0)$$

$$10 \log_{10}(0.5) = 20 \log_{10}(\sqrt{0.5}) \approx -3.01 \text{ dB}$$

$$20 \log_{10}(0.1) = -20 \text{ dB}, 20 \log_{10}(0.01) = -40 \text{ dB}$$

*Complex numbers, radii, angles, unit circle*

$$i \equiv j = \sqrt{-1} = -1/j$$

$$z = x + jy = r e^{j\theta}$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan(y/x) + n\pi, (n = 0, \text{ if } x > 0, n = 1, \text{ if } x < 0)$$

$$x = r \cos(\theta), y = r \sin(\theta)$$

$$e^{j\theta} = \cos(\theta) + j \sin(\theta) \quad (\text{Euler's formula})$$

$$\cos(\theta) = (1/2) \cdot (e^{j\theta} + e^{-j\theta}), \sin(\theta) = (1/2j) \cdot (e^{j\theta} - e^{-j\theta})$$

$$z_1 \cdot z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}, z_1 / z_2 = (r_1 / r_2) e^{j(\theta_1 - \theta_2)}$$

$$|A \cdot B| = |A| \cdot |B|, \angle(A \cdot B) = \angle A + \angle B$$

$$z^n = r^n e^{jn\theta} = r^n (\cos \theta + j \sin \theta)^n = r^n (\cos n\theta + j \sin n\theta)$$

$$z_k = \sqrt[N]{z} = \sqrt[N]{r} e^{j\theta/N} = |\sqrt[N]{r}| e^{j(\theta + 2\pi k)/N}, k = 0, 1, \dots, N - 1$$

*Trigonometric functions*

$$1^\circ = \pi/180 \text{ radians} \approx 0.01745 \text{ rad}, 1 \text{ rad} = 180^\circ/\pi \approx 57.30^\circ$$

$$\text{sinc}(\theta) = \sin(\pi\theta)/(\pi\theta)$$

$$\sin(\theta)/\theta \rightarrow 1, \text{ when } \theta \rightarrow 0; \text{sinc}(\theta) \rightarrow 1, \text{ when } \theta \rightarrow 0$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots + (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \dots \quad (\text{Taylor})$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots + (-1)^n \frac{\theta^{2n}}{(2n)!} + \dots \quad (\text{Taylor})$$

$\theta$	0	$\pi/6$	$\pi/4$	$\pi/3$
$\sin(\theta)$	0	0.5	$\sqrt{2}/2$	$\sqrt{3}/2$
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	0.5
$\theta$	$\pi/2$	$3\pi/4$	$\pi$	$5\pi/4$
$\sin(\theta)$	1	$\sqrt{2}/2$	0	-1
$\cos(\theta)$	0	$-\sqrt{2}/2$	-1	0

$$\pi \approx 3.1416, \sqrt{3}/2 \approx 0.8660, \sqrt{2}/2 \approx 0.7071$$

*Geometric series*

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, |a| < 1$$

$$\sum_{n=0}^N a^n = \frac{1-a^{N+1}}{1-a}, |a| < 1$$

**Continuous-time unit step and unit impulse functions**

$$\mu(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

$$\delta_\Delta(t) = \frac{1}{\Delta} \mu_\Delta(t), \delta(t) = \lim_{\Delta \rightarrow 0} \delta_\Delta(t) \quad (\text{Dirac's delta})$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t - t_0) x(t) dt = x(t_0)$$

In DSP notation  $2\pi\delta(t)$  is computed  $2\pi \int \delta(t) \cdot 1 dt = 2\pi$ , when  $t = 0$ , and = 0 elsewhere.

**Discrete-time unit impulse and unit step functions**

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \quad \mu[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

*Sequence*  $x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k]$ . E.g.:  $x[n] = 2\delta[n + 1] + \delta[n] - \delta[n - 1] = \{2, 1, -1\}$ ,  $x[-1] = 2$ ,  $x[0] = 1$ ,  $x[1] = -1$ .

**Periodic signals**

$$\exists T \in \mathbb{R}_+ : x(t) = x(t + T), \forall t; t, T \in \mathbb{R}$$

$$\exists N \in \mathbb{Z}_+ : x[n] = x[n + N], \forall n; n, N \in \mathbb{Z}$$

*Fundamental period*  $T_0, N_0$  is the smallest  $T > 0, N > 0$ .

$$\text{E.g.: } \cos(\omega n) = \cos(\omega n + 2\pi k), e^{j(\omega n)} = e^{j(\omega n + 2\pi k)}$$

**Convolution**

Convolution is commutative, associative and distributive.

$$y(t) = h(t) \otimes x(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau$$

$$y[n] = h[n] \otimes x[n] = \sum_{k=-\infty}^{\infty} h[k] x[n - k]$$

$$y_C[n] = h[n] \otimes x[n] = \sum_{k=0}^{N-1} h[k] x[n - k > N]$$

**Correlation**

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n] y[n - l] = x[l] \otimes y[-l]$$

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n] x[n - l]$$

**Mean and variance of random signal**

$$m_X = E[X] = \int x p_X(x) dx$$

$$\sigma_X^2 = \int (x - m_X)^2 p_X(x) dx = E[X^2] - m_X^2$$

**Frequencies, angular frequencies, periods**

Here  $f_T$  (sometimes  $f_s$ ) is the sampling frequency.

$$\text{Frequency } f, [f] = \text{Hz} = 1/s.$$

$$\text{Angular frequency } \Omega = 2\pi f = 2\pi/T, [\Omega] = \text{rad/s (analog).}$$

$$\text{Normalized angular frequency } \omega = 2\pi\Omega/\Omega_s = 2\pi f/f_T, [\omega] = \text{rad/sample (digital).}$$

$$\text{Normalized frequency in Matlab } f_{MATLAB} = 2f/f_T, [f_{MATLAB}] = 1/\text{sample.}$$

**Sampling of  $x_a(t)$  by sampling frequency  $f_T$**

$$x_p[n] = x_a(nT) = x_a(n/f_T)$$

$$X_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(j(\Omega - k\Omega_T))$$

**Integral transforms. Properties**

Here all integral transforms share some basic properties.

Examples given with CTFT,  $x[n] \leftrightarrow X(e^{j\omega})$ ,  $x_1[n] \leftrightarrow X_1(e^{j\omega})$ , and  $x_2[n] \leftrightarrow X_2(e^{j\omega})$  are time-domain signals with corresponding transform-domain spectra.  $a$  and  $b$  are constants.

*Linearity.* All transforms are linear.

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

*Time-shifting.* There is a kernel term in transform, e.g.,

$$x[n - k] \leftrightarrow e^{-jk\omega} X(e^{j\omega})$$

*Frequency-shifting.* There is a kernel term in signal e.g.,

$$e^{j\omega_k n} x[n] \leftrightarrow X(e^{j(\omega - \omega_k)})$$

*Conjugate symmetry*  $x^*[n] \leftrightarrow X^*(e^{-j\omega})$ . If  $x[n] \in \mathbb{R}$ , then

$$X(e^{j\omega}) = X^*(e^{-j\omega}), |X(e^{j\omega})| = |X(e^{-j\omega})|, \angle X(e^{j\omega}) = -\angle X(e^{-j\omega}).$$

If  $x[n] \in \mathbb{R}$  and even, then  $X(e^{j\omega}) \in \mathbb{R}$  and even. If  $x[n] \in \mathbb{R}$  and odd, then  $X(e^{j\omega})$  purely  $\in \mathbb{C}$  and odd.

*Time reversal.* Transform variable is reversed, e.g.,

$$x[-n] \leftrightarrow X(e^{-j\omega})$$

*Differentiation.* In time and frequency domain, e.g.,

$$x[n] - x[n - 1] \leftrightarrow (1 - e^{-j\omega}) X(e^{j\omega}), nx[n] \leftrightarrow j \frac{d}{d\omega} X(e^{j\omega})$$

*Duality.* Convolution property convolution in time domain corresponds multiplication in transform domain  $x_1[n] \otimes x_2[n] \leftrightarrow X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$  and multiplication property, vice versa,  $x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega - \theta)}) d\theta$

*Parseval's relation.* Energy in signal and spectral components:

$$\sum |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

**Fourier series of continuous-time periodic signals [ cont. signal  $\leftrightarrow$  discrete frequency ]**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \quad (\text{synthesis})$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\Omega_0 t} dt \quad (\text{analysis})$$

$$x(t - t_0) \leftrightarrow a_k e^{-jk\Omega_0 t_0}$$

$$e^{jM\Omega_0 t} x(t) \leftrightarrow a_{k - M}$$

$$\int_T x_a(\tau) x_b(t - \tau) d\tau \leftrightarrow T a_k b_k$$

$$x_a(t) x_b(t) \leftrightarrow \sum_l a_l b_{k-l}$$

$$\frac{d}{dt} x(t) \leftrightarrow jk\Omega_0 a_k$$

**Continuous-time Fourier-transform (CTFT) [ cont. signal  $\leftrightarrow$  cont. frequency ]**

$$\begin{aligned}
 x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \text{ (synthesis)} \\
 X(j\Omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \text{ (analysis)} \\
 x(t - t_k) &\leftrightarrow e^{j\Omega t_k} X(j\Omega) \\
 e^{j\Omega_k t} x(t) &\leftrightarrow X(j(\Omega - \Omega_k)) \\
 x_a(t) \otimes x_b(t) &\leftrightarrow X_a(j\Omega) X_b(j\Omega) \\
 x_a(t)x_b(t) &\leftrightarrow \frac{1}{2\pi} X_a(j\Omega) \otimes X_b(j\Omega) \\
 \frac{d}{dt} x(t) &\leftrightarrow j\Omega X(j\Omega) \\
 t x(t) &\leftrightarrow j \frac{d}{d\Omega} X(j\Omega) \\
 e^{j\Omega_0 t} &\leftrightarrow 2\pi \delta(\Omega - \Omega_0) \\
 \cos(\Omega_0 t) &\leftrightarrow \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)] \\
 \sin(\Omega_0 t) &\leftrightarrow j\pi[\delta(\Omega + \Omega_0) - \delta(\Omega - \Omega_0)] \\
 x(t) = 1 &\leftrightarrow 2\pi \delta(\Omega) \\
 x(t) &= \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \leftrightarrow \frac{2 \sin(\Omega T_1)}{\Omega} \\
 \frac{\sin(Wt)}{\pi t} &\leftrightarrow X(j\Omega) = \begin{cases} 1, & |\Omega| < W \\ 0, & |\Omega| > W \end{cases} \\
 \delta(t) &\leftrightarrow 1 \\
 \delta(t - t_k) &\leftrightarrow e^{j\Omega t_k} \\
 e^{-at} \mu(t) &\leftrightarrow \frac{1}{a + j\Omega}, \text{ where } \mathcal{R}\{a\} > 0
 \end{aligned}$$

**Discrete-time Fourier-transform (DTFT)** [ discrete signal  $\leftrightarrow$  cont. frequency ]

$$\begin{aligned}
 x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \text{ (synthesis)} \\
 X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \text{ periodic with } 2\pi \text{ (analysis)} \\
 a x[n - k] &\leftrightarrow a e^{-jk\omega} X(e^{j\omega}) \\
 e^{j\omega_k n} x[n] &\leftrightarrow X(e^{j(\omega - \omega_k)}) \\
 x_1[n] \otimes x_2[n] &\leftrightarrow X_1(e^{j\omega}) \cdot X_2(e^{j\omega}) \\
 x_1[n] \cdot x_2[n] &\leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta}) X_2(e^{j(\omega - \theta)}) d\theta \\
 n x[n] &\leftrightarrow j \frac{d}{d\omega} X(e^{j\omega}) \\
 e^{j\omega_0 n} &\leftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l) \\
 \cos(\omega_0 n) &\leftrightarrow \pi \sum_{l=-\infty}^{\infty} [\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)] \\
 \sin(\omega_0 n) &\leftrightarrow j\pi \sum_{l=-\infty}^{\infty} [\delta(\omega + \omega_0 - 2\pi l) - \delta(\omega - \omega_0 - 2\pi l)] \\
 x[n] = 1 &\leftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) \\
 x[n] &= \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases} \leftrightarrow \frac{\sin(\omega(N_1 + 0.5))}{\sin(\omega/2)}
 \end{aligned}$$

$$\frac{\sin(Wn)}{\pi n} = \frac{W}{\pi} \text{sinc}\left(\frac{Wn}{\pi}\right) \leftrightarrow X(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq W \\ 0, & W < |\omega| \leq \pi \end{cases}$$

$$\begin{aligned}
 \delta[n] &\leftrightarrow 1 \\
 a \delta[n - k] &\leftrightarrow a e^{-jk\omega} \\
 a^n \mu[n] &\leftrightarrow \frac{1}{1 - a e^{-j\omega}}, \quad |a| < 1
 \end{aligned}$$

**N-point Discrete Fourier-transform (DFT)** [ discrete signal  $\leftrightarrow$  discrete frequency, "discrete-time Fourier series" ]

Connection to DTFT:  $X[k] = X(e^{j\omega})|_{\omega=2\pi k/N}$

$$\begin{aligned}
 W_N &= e^{-j2\pi/N} \\
 x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \leq n \leq N - 1 \text{ (synthesis)} \\
 X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \leq k \leq N - 1 \text{ (analysis)} \\
 x < n - n_0 > N &\leftrightarrow W_N^{-kn_0} X[k] \\
 W_N^{-kn_0} x[n] &\leftrightarrow X < k - k_0 > N \\
 y_C[n] &= h[n] \otimes x[n] \leftrightarrow H[k] \cdot X[k] = Y[k]
 \end{aligned}$$

**Laplace transform:**

Convergence with a certain ROC (region of convergence). Connection to continuous-time Fourier-transform:  $s = j\Omega$

$$\begin{aligned}
 x(t) &= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds \text{ (synthesis)} \\
 X(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt \text{ (analysis)}
 \end{aligned}$$

**z-transform:**

Convergence with a certain ROC (region of convergence). Connection to discrete-time Fourier-transform:  $z = e^{j\omega}$

$$\begin{aligned}
 x[n] &= \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz, \quad C \text{ in ROC of } X(z) \text{ (synthesis)} \\
 X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \text{ (analysis)} \\
 a x[n - k] &\leftrightarrow a z^{-k} X(z) \\
 x_1[n] \otimes x_2[n] &\leftrightarrow X_1(z) \cdot X_2(z) \\
 \delta[n] &\leftrightarrow 1, \quad \text{ROC all } z \\
 a \delta[n - k] &\leftrightarrow a z^{-k}, \quad \text{all } z, \text{ except } 0 \text{ (} k > 0 \text{) or } \infty \text{ (} k < 0 \text{)} \\
 \mu[n] &\leftrightarrow \frac{1}{1 - z^{-1}}, \quad |z| > 1 \\
 -\mu[-n - 1] &\leftrightarrow \frac{1}{1 - z^{-1}}, \quad |z| < 1 \\
 a^n \mu[n] &\leftrightarrow \frac{1}{1 - az^{-1}}, \quad |z| > |a|
 \end{aligned}$$

$$\begin{aligned}
 na^n \mu[n] &\leftrightarrow \frac{az^{-1}}{(1 - az^{-1})^2}, \quad |z| > |a| \\
 (n + 1)a^n \mu[n] &\leftrightarrow \frac{1}{(1 - az^{-1})^2}, \quad |z| > |a| \\
 r^n \cos(\omega_0 n) \mu[n] &\leftrightarrow \frac{1 - r \cos(\omega_0) z^{-1}}{1 - 2r \cos(\omega_0) z^{-1} + r^2 z^{-2}}, \quad |z| > |r| \\
 r^n \sin(\omega_0 n) \mu[n] &\leftrightarrow \frac{r \sin(\omega_0) z^{-1}}{1 - 2r \cos(\omega_0) z^{-1} + r^2 z^{-2}}, \quad |z| > |r|
 \end{aligned}$$

**LTI filter analysis**

Stability  $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ ; unit circle belongs to ROC  
 Causality  $h[n] = 0, n < 0$ ;  $\infty$  belongs to ROC  
 Unit step response  $s[n] = \sum_{k=-\infty}^n h[k]$   
 Causal transfer function of order  $\max\{M, N\}$   
 $H(z) = B(z)/A(z) = K \cdot \frac{\sum_{m=0}^M b_m z^{-m}}{\sum_{n=0}^N a_n z^{-n}} = G \cdot \frac{\prod_{m=1}^M (1 - d_m z^{-1})}{\prod_{n=1}^N (1 - p_n z^{-1})}$   
 where zeros  $d_m$ ;  $B(z) = 0$ ; and poles  $p_n$ ;  $A(z) = 0$   
 Frequency response  $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$   
 Frequency response vs  $H(z)$  vs  $H[k]$   
 $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$ ;  $H[k] = H(e^{j\omega})|_{\omega=2\pi k/N}$   
 Magnitude response, phase response  $H(e^{j\omega}) = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$   
 Group delay  $\tau(\omega) = -\frac{d}{d\omega} \angle H(e^{j\omega})$   
 Four types of linear-phase FIR filters,  $h[n] = h[N - 1 - n]$  (even/odd symmetric), or  $h[n] = -h[N - 1 - n]$  (e/o antis.).  
 Zeros symmetric w.r.t. unit circle:  $r e^{\pm j\theta}$  and  $(1/r) e^{\mp j\theta}$ .

Important transform pairs and properties:

$$\begin{aligned}
 a \delta[n - k] &\leftrightarrow a e^{-jk\omega} \leftrightarrow a z^{-k} \\
 a^n \mu[n] &\leftrightarrow 1/[1 - a e^{-j\omega}] \leftrightarrow 1/[1 - a z^{-1}] \\
 a x[n - k] &\leftrightarrow a e^{-jk\omega} X(e^{j\omega}) \leftrightarrow a z^{-k} X(z) \\
 K \cdot \sum_{m=M_1}^{M_2} b_m x[n - m] &= \sum_{k=K_1}^{K_2} a_k y[n - k] \leftrightarrow \dots \\
 \dots K \cdot \frac{\sum_{m=M_1}^{M_2} b_m z^{-m}}{\sum_{k=K_1}^{K_2} a_k z^{-k}} &\leftrightarrow K \cdot \frac{\sum_{m=M_1}^{M_2} b_m e^{-jm\omega}}{\sum_{k=K_1}^{K_2} a_k e^{-jk\omega}} \\
 y[n] = h[n] \otimes x[n] &\leftrightarrow Y(z) = H(z) \cdot X(z) \\
 \text{rectangle} &\leftrightarrow \text{sinc}, \text{ sinc} \leftrightarrow \text{rectangle}
 \end{aligned}$$

**LTI filter design (synthesis)**

Bilinear transform  $H(z) = H(s)|_s$  and prewarping  
 $s = k \cdot (1 - z^{-1})/(1 + z^{-1})$ ,  $k = 1$  or  $k = 2/T = 2f_T$   
 $\Omega_{\text{prewarp},c} = k \cdot \tan(\omega_c/2)$ ,  $k = 1$  or  $k = 2/T = 2f_T$   
 Spectral transformations,  $\hat{\omega}_c$  desired cut-off  
 LP-LP  $z^{-1} = (\hat{z}^{-1} - \alpha)/(1 - \alpha \hat{z}^{-1})$ , where  
 $\alpha = \sin(0.5(\omega_c - \hat{\omega}_c))/\sin(0.5(\omega_c + \hat{\omega}_c))$   
 LP-HP  $z^{-1} = -(\hat{z}^{-1} + \alpha)/(1 + \alpha \hat{z}^{-1})$ , where  
 $\alpha = -\cos(0.5(\omega_c + \hat{\omega}_c))/\cos(0.5(\omega_c - \hat{\omega}_c))$

Windowed Fourier series method

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| \geq \omega_c \end{cases} \leftrightarrow h[n] = \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c n}{\pi}\right)$$

$$\begin{aligned}
 h_{FIR}[n] &= h_{\text{ideal}}[n] \cdot w[n] \\
 H_{FIR}(e^{j\omega}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{\text{ideal}}(e^{j\theta}) W(e^{j(\omega - \theta)}) d\theta \\
 \text{Fixed window functions, order } N = 2M, -M \leq n \leq M: & \\
 \text{Rectangular } w[n] &= 1 \\
 \text{Hamming } w[n] &= 0.54 + 0.46 \cos((2\pi n)/(2M)) \\
 \text{Hann } w[n] &= 0.5 \cdot (1 + \cos((2\pi n)/(2M))) \\
 \text{Blackman } w[n] &= 0.42 + 0.5 \cos\left(\frac{2\pi n}{2M}\right) + 0.08 \cos\left(\frac{4\pi n}{2M}\right) \\
 \text{Bartlett } w[n] &= 1 - (|n|/M)
 \end{aligned}$$

**Implementation**

Radix-2 DIT N-FFT butterfly equations,  $L\{x[n]\} = N$

$$\begin{cases} \Psi_{r+1}[\alpha] = \Psi_r[\alpha] + W_{N_r}^l \Psi_r[\beta] \\ \Psi_{r+1}[\beta] = \Psi_r[\alpha] - W_{N_r}^l \Psi_r[\beta] \end{cases}$$

Inputs  $x[n]$  in bit-reversed order in  $\Psi_1$ , outputs in  $\Psi_R$ ,  $R = \log_2 N + 1$ ; levels  $\Psi_1, \dots, \Psi_R$ ,  $r \in [1, R - 1]$ ; multipliers  $W_{N_r} = e^{-j2\pi/N_r}$ ,  $N_r = 2^r$ ,  $l \in [0, 2^{r-1} - 1]$ .

**Multirate systems**

Upsampling (interpolation) with factor  $L$ ,  $\uparrow L$

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ x_u[n] = 0, & \text{otherwise} \end{cases}$$

$$X_u(z) = X(z^L), \quad X_u(e^{j\omega}) = X(e^{j\omega L})$$

Downsampling (decimation) with factor  $M$ ,  $\downarrow M$

$$\begin{aligned}
 x_d[n] &= x[nM] \\
 X_d(z) &= (1/M) \sum_{k=0}^{M-1} X(z^{1/M} W_M^{z^{-k}}) \\
 X_d(e^{j\omega}) &= (1/M) \sum_{k=0}^{M-1} X(e^{j(\omega - 2\pi k)/M})
 \end{aligned}$$